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PROBLEMS IN EXTREMAL COMBINATORICS

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PREFACE

This thesis is based on the following twelve research articles, nine of which are at the time of writing published in journals, two are accepted for publication and one is preliminarily accepted, subject to minor revision (based on three referee reports).

All these works are motivated by extremal problems in combinatorics and all the main results are listed and described in the Introduction. In these publications from years between 2016 and 2022, we solve several open problems posed by various researchers and we significantly improve previously known results.

For coherence we have adjusted the wording slightly. For instance where in the original manuscripts it said “in this paper", here we write “in this chapter". Similarly, some section names have been adjusted and duplicates in the notation removed. The spelling has been changed to British throughout. No changes to the actual content of the papers have been made.

[AM21] A. Shapira and M. Tyomkyn. “A Ramsey variant of the Brown-Erdos-Sos conjecture.” In: *Bulletin of the London Mathematical Society* 53.5 (2021), pp. 1453–1469.

[AM22] A. Shapira and M. Tyomkyn. *Weakly saturated hypergraphs and a conjecture of Tuza*. To appear in Proceedings of the AMS. 2022.

[DM21a] M. Tancer D. Bulavka and M. Tyomkyn. *Weak saturation of multipartite hypergraphs*. Accepted to Combinatorica, subject to minor revision. Preprint available at arxiv:2109.03703. 2021.

[DM21b] D. Conlon and M. Tyomkyn. “Repeated patterns in proper edge-colourings.” In: *SIAM J. Discrete Math* 35.3 (2021), pp. 2249–2264.

[DM23] D. Conlon and M. Tyomkyn. “Ramsey numbers of trails and circuits.” In: *Journal of Graph Theory* 102.1 (2023), pp. 194–196.

[DM22] S. Luo D. Conlon and M. Tyomkyn. *Monochromatic components with many edges*. Accepted to Journal of Combinatorics. Preprint available at arxiv:2204.11360. 2022.

[DM16] D. Hefetz and M. Tyomkyn. “Universality of graphs with few triangles and anti-triangles.” In: *Combinatorics, Probability and Computing* 25.4 (2016), pp. 560–576.

[DM18] D. Hefetz and M. Tyomkyn. “On the inducibility of cycles.” In: *J. Combinatorial Theory, Ser. B* 133 (2018), pp. 243–258.

[MM21] A. Shapira M. Amir and M. Tyomkyn. “Two Erdos–Hajnal-type theorems in hypergraphs.” In: *J. Combinatorial Theory, Ser. B* 146 (2021), pp. 417–438.

[M T17] M. Tyomkyn. “Lagrangians of hypergraphs: The Frankl-Furedi conjecture holds almost everywhere.” In: *Journal of the London Mathematical Society* 96.3 (2017), pp. 584–600.

[M T21] M. Tyomkyn. “Many disjoint triangles in co-triangle-free graphs.” In: *Combinatorics, Probability and Computing* 30.1 (2021), pp. 153–162.

[NM20] M. Krivelevich N. Alon D. Hefetz and M. Tyomkyn. “Edge-statistics on large graphs.” In: *Combinatorics, Probability and Computing* 29.2 (2020), pp. 163–189.

CONTENTS

1 INTRODUCTION

1.1 PRELIMINARIES

Extremal Combinatorics studies interactions between different properties of a discrete object, for instance, how many edges in a graph enforce the existence of certain subgraphs. Its study was initiated in the 1930s by the Hungarian mathematical school led by Turán, Erdős and others, and nowadays extremal problems constitute one of the most important branches of research in combinatorics, with applications to additive number theory, discrete geometry and theoretical computer science, to name a few. Their importance within mathematics was acknowledged by the Abel Prize committee in 2012, when the prize was awarded to E. Szemerédi. Likewise, the methods and proof techniques have evolved over the years from ad hoc induction and counting arguments to intricate uses of tools from Algebra, Analysis and Probability.

These developments are reflected in the present thesis. It is comprised of seven chapters (excluding the introduction) covering eleven of my research papers published or accepted for publication between 2016 and 2022, and one paper which is submitted at the time of writing.

Besides various combinatorial and graph theoretic arguments in our proofs we use a number of techniques from multivariate optimization (Chapters 2,4 and 6) probability (Chapters 3,4,5 and 7) and algebra (Chapters 7 and 8). We therefore believe this thesis presents a good cross-section of modern extremal combinatorics and its interplay with other areas of mathematics.

In the time since our papers were published, a number of our results were extended by various groups of researchers, including several leaders of the field. In particular, extending our work in Chapter 2, Gruslys, Letzter and Morrison [VSN20, VSN] proved the Frankl-Füredi conjecture for large n for $r = 3$ and disproved it for all $r \geq 4$. Keevash and Long [PJ] established a density counterpart to our Ramsey-type result on the Brown-Erdős-Sós conjecture from Chapter 3. Our bound on inducibility of cycles in Chapter 4 was subsequently improved by Král', Norin and Volec [KNV19]. Perhaps the most spectacular development was a proof of our Edge-statistics conjecture of Chapter 4 by three groups of researchers. Kwan, Tran and Sudakov [KST19] settled it in the superlinear regime (that is, when $\ell = \omega(k)$), while Fox and Sauermann [JL20] and, independently, Martinsson, Mousset, Noever and Trujić [Mar+19] solved it for the remaining values of ℓ. Our results in the second part of Chapter 6 were generalized by Gruslys and Letzter [VS20a, VS20b], who proved the Erdős conjecture discussed there in full. Finally, a number of researchers including most significantly O. Janzer [O J] made progress on some of the open problems we raised in Chapter 7.

In the subsequent sections of the Introduction we shall describe the content of the respective chapters of this thesis.

1.2 LAGRANGIANS OF HYPERGRAPHS

In Chapter 2 we deal with the problem of Lagrangians of hypergraphs. It is based on the solo paper [M T17].

Multilinear polynomials are of central interest in most branches of modern mathematics, and extremal combinatorics is by no means an exception. In particular, a large number of hypergraph Turán problems reduce to calculating or estimating the Lagrangian of a hypergraph, which is a constrained maximum of the multilinear function naturally associated with the hypergraph.

To set the scene, we need a few definitions. We follow standard notation of extremal combinatorics (see e.g. [Bol86]). In particular, for $n, r \in \mathbb{N}$, we write $[n]$ for the set $\{1, \ldots, n\}$ and, given a set X, by $X^{(r)}$ we denote the set family $\{A \subseteq X : |A| = r\}$. Dealing with finite families of finite sets we will be freely switching between the set system and the hypergraph points of view: with no loss of generality, we can assume our hypergraphs to be defined on $\mathbb{N}$, yet we write $e(H)$ for the number of sets ('edges') in H.

For a finite r-uniform hypergraph $H \subseteq [n]^{(r)}$ and a vector of real numbers (referred as a *weighting*) $\vec{y} := (y_1, \ldots, y_n)$ consider a multilinear polynomial function

$$L(H, \vec{y}) := \sum_{A \in H} \prod_{i \in A} y_i.$$

The *Lagrangian* of H is defined as its maximum on the standard simplex

$$\lambda(H) := \max\{L(H, \vec{y}) \colon y_1, \ldots, y_n \geq 0; \sum_{i=1}^{n} y_i = 1\};$$

note that, by compactness, the maximum does always exist (but need not be unique).

The above notion was introduced in 1965 by Motzkin and Strauss [TE65] for $r = 2$, that is for graphs, in order to give a new proof of Turán's theorem. Later it was extended to uniform hypergraphs, where the Lagrangian plays an important role in governing densities of blow-ups. In particular, using Lagrangians of r-graphs, Frankl and Rödl [PV84] disproved a conjecture of Erdős [Erd83] by exhibiting infinitely many non-jumps for hypergraph Turán densities. In the following years the Lagrangian has found numerous applications in hypergraph Turán problems; for more details we refer to a survey by Keevash [Kee11] and the references therein. Further results, which appeared after the publication of [Kee11], include [DP13] and [Q T+16].

In this chapter we address the problem of maximising the Lagrangian itself over all r-graphs with a fixed number of edges. Let $H^{m,r}$ be the subgraph of $\mathbb{N}^{(r)}$ consisting of the first m sets in the colexicographic order (recall that this is the ordering on $\mathbb{N}^{(r)}$ in which $A < B$ if $\max(A \triangle B) \in B$). In 1989 Frankl and Füredi [PZ89] conjectured that the maximum Lagrangian of an r-graph on m edges is realised by $H^{m,r}$.

Conjecture 1.2.1 ([PZ89])**.** $\lambda(H^{m,r}) = \max\{\lambda(H)\colon H \subseteq \mathbb{N}^{(r)}, e(H) = m\}$.

In an important special case, which we refer to as the *principal case*, Conjecture 1.2.1 states that for $m = \binom{t}{r}$ the maximum Lagrangian is attained on $H^{m,r} = [t]^{(r)}$, where we have $\lambda(H^{m,r}) = \lambda([t]^{(r)}) = \frac{1}{t^r}\binom{t}{r}$. While initially the Frankl-Füredi conjecture was motivated by applications to hypergraph Turán problems, we think it also interesting in its own right, as it makes a natural and general statement about maxima of multilinear functions.

For $r = 2$ the validity of Conjecture 1.2.1 is easy to see and follows from the arguments of Motzkin and Strauss [TE65]. In fact, the Lagrangian of a graph H is attained by equi-distributing the weights between the vertices of the largest clique of H, resulting in $\lambda(H) = \frac{\omega(H)-1}{2\omega(H)}$. Since $H^{m,r}$ has the largest clique size over all graphs on m edges, Conjecture 1.2.1 holds.

On the other hand, the situation for hypergraphs is far more complex, since for $r \geq 3$, unlike in the graph case, no direct way of inferring $\lambda(H)$ from the structure of H is known. Hence one is confined to estimating the Lagrangians of different r-graphs against each other without calculating them directly.

For $r = 3$ Talbot [J T02] proved that Conjecture 1.2.1 holds whenever $\binom{t-1}{3} \leq m \leq \binom{t-1}{3} + \binom{t-2}{2} - (t-1) = \binom{t}{3} - (2t-3)$ for some $t \in \mathbb{N}$. Note that this range covers an asymptotic density 1 subset of $\mathbb{N}$, and also includes the principal case $m = \binom{t-1}{3}$. Recently Tang, Peng, Zhang and Zhao [Q T+16] extended the above range to $\binom{t-1}{3} \leq m \leq \binom{t-1}{3} + \binom{t-2}{2} - \frac{1}{2}(t-1)$. Furthermore, Conjecture 1.2.1 is known to hold when $\binom{t}{3} - m$ is a small constant, but for the remaining values of m it is still open.

In contrast to this, for $r \geq 4$ much less has been known so far, as Talbot's proof method for $r = 3$, perhaps surprisingly, does not immediately transfer. Talbot showed in the same paper [J T02] that for every $r \geq 4$ there is a constant $\gamma_r > 0$ such that if $\binom{t-1}{r} \leq m \leq \binom{t}{r} - \gamma_r t^{r-2}$ and H is supported on t vertices (that is, ignoring isolated vertices, H is a subgraph of $[t]^{(r)}$), then indeed $\lambda(H) \leq \lambda(H^{m,r})$. Still, for no value of m, apart from some trivial ones, Conjecture 1.2.1 has been known to hold. Our main goal in this chapter is to close this gap by confirming the Frankl-Füredi Conjecture for 'most' values of m for any given $r \geq 4$, including the principal case for large m.

Theorem 1.2.2. *For every $r \geq 4$ there exists $\gamma_r > 0$ such that for all $\binom{t-1}{r} \leq m \leq \binom{t}{r} - \gamma_r t^{r-2}$ we have*

$$\lambda(H^{m,r}) = \max\{\lambda(H) \colon H \subseteq \mathbb{N}^{(r)}, e(H) = m\}.$$

Corollary 1.2.3. *For every $r \geq 4$ there exists $t_r \in \mathbb{N}$ such that for all $t \in \mathbb{N}$ with $t \geq t_r$ we have*

$$\max\left\{\lambda(H) \colon H \subseteq \mathbb{N}^{(r)}, e(H) = \binom{t}{r}\right\} = \lambda([t]^{(r)}) = \frac{1}{t^r}\binom{t}{r}.$$

By monotonicity, we obtain another immediate corollary, which can be viewed as a strong approximate version of Conjecture 1.2.1.

Corollary 1.2.4. *For every $r \geq 4$ there exists $t_r \in \mathbb{N}$ such that for all $t \geq t_r$ the following holds. Suppose that $\binom{t-1}{r} < m \leq \binom{t}{r}$ and that H is an r-graph with $e(H) = m$. Then*

$$\lambda(H) \leq \frac{1}{t^r}\binom{t}{r}.$$

When H is supported on $[t]$ we give a proof of a stronger statement, namely that in this case we can take $\gamma_r = (1 + o(1))/(r-2)!$ in Theorem 1.2.2. More precisely, we claim the following.

Theorem 1.2.5. *For every $r \geq 3$ there exists a constant $\delta_r > 0$ such that for all $\binom{t-1}{r} \leq m \leq \binom{t}{r} - \binom{t-2}{r-2} - \delta_r t^{r-9/4}$ we have*

$$\lambda(H^{m,r}) = \max\{\lambda(H) \colon H \subseteq [t]^{(r)}, e(H) = m\}.$$

For $r = 3$ it was implicitly shown by Talbot in [J T02] that for any $\binom{t-1}{3} < m \leq \binom{t}{3}$, that is for all $m \in \mathbb{N}$, the 3-graph maximising the Lagrangian amongst all m-edge 3-graphs can be assumed to be supported on $[t]$. Combined with Theorem 1.2.5, this yields, for large m, an improvement of the bounds in [J T02] and [Q T+16].

Corollary 1.2.6. *There exists a constant $\delta_3 > 0$ such that for all $\binom{t-1}{3} \leq m \leq \binom{t}{3} - (t-2) - \delta_3 t^{3/4}$ we have*

$$\lambda(H^{m,r}) = \max\{\lambda(H) \colon H \subseteq \mathbb{N}^{(r)}, e(H) = m\}.$$

Overview of the proof

The proof of our main result, Theorem 1.2.2, uses a number of well-known properties of the Lagrangian, as well as induction on r and some facts about uniform set systems such as the Kruskal-Katona theorem. We begin by considering the r-graph $G \subseteq \mathbb{N}^{(r)}$ and the weighting $\vec{x}$ that (co-)achieve the largest Lagrangian amongst all r-graphs of size m. We assume that $\lambda(G)$ is strictly larger than $\lambda(H^{m,r})$, and aim to show that then m must lie outside the range specified in Theorem 1.2.2. This is carried out as follows.

First, we make a number of standard assumptions on G and $\vec{x}$. We assume that G covers pairs (meaning that any two vertices of G are contained in some edge) and has the minimum possible number of vertices (referred as the 'support'). Assuming by symmetry that the entries of $\vec{x}$ are listed in descending order, we can also claim that G is left-compressed. As a consequence, we obtain some bounds on the sizes of link hypergraphs of G, as stated in Proposition 2.2.1 (which is where we need the Kruskal-Katona theorem).

It turns out that the parameter we are most interested in is T, the support of G. By combining Proposition 2.2.8, which is a standard tool that relates $\lambda(G)$ to the Lagrangians of its link hypergraphs, with a number of further ideas such as induction of r and Proposition 2.2.1, we gradually establish better and better bounds on T as well as on related parameters such as x_1 and x_T (the largest and the smallest entries of $\vec{x}$). This part of the argument culminates in Lemma 2.5.1 and Lemma 2.6.2, where we show, respectively, that $T = t + C$ and $x_1 < 2x_{t-3\alpha}$ for some constants $0 \leq C \leq C_0(r)$ and $\alpha = \alpha(r)$.

In the final part of the proof the above bounds are applied to replace a number of 'bad' edges of G with some 'good' edges from $[t]^{(r)} \setminus G$ such that the resulting graph does G' not cover any pair in $\{t-1, \dots, T\}^{(2)}$, thus $\lambda(G') \leq \lambda(H^{m,r}) < \lambda(G)$. The estimates on T and x_1 ensure that the good edges are reasonably heavy, so that, unless $\binom{t}{r} - m$ is small (in which case we might not find enough good edges), the total weight of the good edges is greater than that of the bad edges, resulting in $\lambda(G') > \lambda(G)$, a contradiction. Hence, $\binom{t}{r} - m$ has to be small, completing the proof.

1.3 THREE RAMSEY RESULTS FOR UNIFORM HYPERGRAPHS

The work in this chapter is based on papers [MM21] (joint with M. Amir and A. Shapira) and [AM21] (with A. Shapira).

Let us say that a set of vertices in a graph (or hypergraph) is *homogeneous* if it spans either a clique (i.e. a complete graph) or an independent set (i.e. an empty graph). Ramsey's theorem states that every graph contains a homogeneous set of size $\frac{1}{2}\log_2 n$, and Erdős proved that in general, one cannot expect to find a homogeneous set of size larger than $2\log_2 n$ (see [GRS91]). Since Erdős's example uses random graphs, and random graphs are universal (with high probability), that is, they contain an induced copy of every fixed graph H, it is natural to ask what happens if we assume that G is non-universal, or equivalently, that it is induced H-free for some fixed H. A theorem of Erdős and Hajnal [PA89] states that in this case we are guaranteed to have a homogeneous set of size $2^{\Omega(\sqrt{\log n})}$, that is, a significantly larger set than in the worst case. The notorious Erdős–Hajnal Conjecture states that one should be able to go even further and improve this bound to n^c, where $c = c(H)$. We refer the reader to [M C14] for more background on this conjecture and related results.

Conlon, Fox and Sudakov [DJB12] and Rödl and Schacht [VS12] have recently initiated the study of problems of this type in the setting of r-uniform hypergraphs (or r-graphs for short). Our first aim in this chapter is to obtain two results of this flavour described below.

Almost homogeneous sets in non-universal hypergraphs

Our first result is motivated by a theorem of Rödl [V R86]. Let us say that a set of vertices W in a graph is η-homogeneous if W either contains at least $(1-\eta)\binom{|W|}{2}$ or at most $\eta\binom{|W|}{2}$ edges. It is a standard observation that Erdős's lower bound for Ramsey's theorem (mentioned above), actually shows that some (actually, most) graphs of order n do not even contain $\frac{1}{4}$-homogeneous[1] sets of size $O(\log n)$. In other words, in the worst case relaxing 0-homogeneity to $\frac{1}{4}$-homogeneity does not make the problem easier. Rödl's [V R86] surprising theorem then states that if G is non-universal then for any $\eta > 0$, it contains an η-homogeneous set of size $\Omega(n)$, where the hidden constant depends on η. Fox and Sudakov [JB08] gave a new proof of Rödl's theorem, which does not rely on Szemerédi's regularity lemma, and therefore provides a much better bound on the implicit constant in $\Omega(n)$.

It is natural to ask if a similar[2] result holds also in hypergraphs. Random 3-graphs show that, in the worst case, the largest $\frac{1}{4}$-homogeneous set in a 3-graph might be of size $O(\sqrt{\log n})$, and a matching lower bound of $\Omega(\sqrt{\log n})$ was proved by Conlon, Fox and Sudakov [DJB11]. Our first theorem in this chapter shows that, as in graphs, if we assume that a 3-graph is non-universal then we can find a much larger almost homogeneous set.

Theorem 1.3.1. *For every 3-graph $\mathcal{F}$ and $\eta > 0$ there is $c = c(\mathcal{F}, \eta) > 0$ such that every induced $\mathcal{F}$-free 3-graph on n vertices contains an η-homogeneous set of size $c \log n$.*

1 One can easily replace the $\frac{1}{4}$ with any constant smaller than $\frac{1}{2}$. We will stick with the $\frac{1}{4}$ in order to streamline the presentation.

2 We of course say that a set of vertices W in an r-graph is η-homogeneous if W either contains at least $(1-\eta)\binom{|W|}{r}$ or at most $\eta\binom{|W|}{r}$ edges.

Rödl [V R86] found an example of a non-universal 3-graph in which the largest $\frac{1}{4}$-homogeneous set has size $O(\log n)$.[3] Hence, the bound in Theorem 1.3.1 is tight up to the constant c. We will describe in Section 3.5 (see Proposition 3.5.2) a generalization of Rödl's example, giving for every $r \geq 3$ an example of a non-universal r-graph in which the size of the largest $\frac{1}{4}$-homogeneous set is $O((\log n)^{1/(r-2)})$. It seems reasonable to conjecture that this upper bound is tight, that is, that for every $r \geq 3$ every non-universal r-graph has an almost homogeneous set of size $\Omega((\log n)^{1/(r-2)})$.

Let K_k denote the complete graph on k vertices and let $K_k^{(3)}$ denote the complete 3-graph on k vertices. It is easy to see that up to a change of constants, a set of vertices has edge density close to $0/1$ (i.e is η-homogeneous for some small η), if and only if it has K_k-density close to 1 either in the graph or in its complement. The same applies to 3-graphs. An interesting feature of the proof of Theorem 1.3.1 is that instead of gradually building a set of vertices with very large/small edge density, we find it easier to build such a set with large $K_k^{(3)}$-density either in $\mathcal{G}$ or its complement. The way we gradually build such a set is by applying a variant of a greedy embedding scheme used by Rödl and Schacht [VS12] in order to give an alternative proof of an elegant theorem of Nikiforov [V No8] (this alternative proof is also implicit in [DJB12]). To get this embedding scheme 'started' we prove a lemma saying that if a 3-graph $\mathcal{G}$ is non-universal then there is a graph G on a subset of $V(\mathcal{G})$ such that either almost all or almost none of the K_k's of G are also $K_k^{(3)}$'s in $\mathcal{G}$. This latter statement is proved via the hypergraph regularity method.

Complete partite sets in non-universal hypergraphs

Determining the size of the largest homogeneous set in a 3-graph is still a major open problem, see [DJB10]. The best known lower and upper bounds are of order $\log\log n$ and $\sqrt{\log n}$ respectively. It is thus hard to formulate a 3-graph analogue of the Erdős–Hajnal Theorem since it is not clear which bound one is trying to beat. At any rate, as of now, we do not even know if a non-universal 3-graph contains a homogeneous set of size $\omega(\log\log n)$ (see Section 3.5 for further discussion on this problem). This motivated the authors of [DJB12, VS12] to look at the following related problem. Let $K_{t,t,t}^{(3)}$ denote the complete 3-partite 3-graph with each part of size t. It is a well known fact [P E64] that every 3-graph of positive density contains a copy of $K_{t,t,t}^{(3)}$ with $t = \Omega(\sqrt{\log n})$. This immediately means that for every 3-graph $\mathcal{G}$, either $\mathcal{G}$ or its complement contains a $K_{t,t,t}^{(3)}$ with $t = \Omega(\sqrt{\log n})$. As evidenced by random 3-graphs, this bound is tight. A natural question,

3 The $O(\log n)$ bound is implicit in [V R86], and was first mentioned explicitly in [DJB12].

which was first addressed by Conlon, Fox and Sudakov [DJB12] and by Rödl and Schacht [VS12] is whether one can improve upon this bound when $\mathcal{G}$ is assumed to be non-universal.

It will be more convenient to switch gears at this point, and let $R_{3,\mathcal{F}}(t)$ denote the size of the largest induced $\mathcal{F}$-free 3-graph $\mathcal{G}$, so that neither $\mathcal{G}$ nor $\overline{\mathcal{G}}$ contain a copy of $K^{(3)}_{t,t,t}$. So the question posed at the end of the previous paragraph is equivalent to asking if for every fixed $\mathcal{F}$ we have $R_{3,\mathcal{F}}(t) \leq 2^{o(t^2)}$, and the results of [DJB12, VS12] establish that this is indeed the case. [4] Conlon, Fox and Sudakov [DJB12] also found an example of a 3-graph $\mathcal{F}$ for which $R_{3,\mathcal{F}}(t) \geq 2^{\Omega(t)}$. Our second result improves their lower bound as follows.

Theorem 1.3.2. *There is a 3-graph $\mathcal{F}$ for which $R_{3,\mathcal{F}}(t) \geq t^{\Omega(t)}$.*

As discussed in [DJB12], it is natural to consider the corresponding problem in general r-graphs. Letting $K^{(r)}_{t,\ldots,t}$ denote the complete r-partite r-graph with parts of size t, we define $R_{r,\mathcal{F}}(t)$ to be the size of the largest induced $\mathcal{F}$-free r-graph $\mathcal{G}$, so that neither $\mathcal{G}$ nor $\overline{\mathcal{G}}$ contain a copy of $K^{(r)}_{t,\ldots,t}$. It follows from [P E64], which establishes that in *every* r-graph $\mathcal{G}$ on $2^{\Omega(t^{r-1})}$ vertices with density $1/2$ we can find a $K^{(r)}_{t,\ldots,t}$, that

$$R_{r,\mathcal{F}}(t) \leq 2^{O(t^{r-1})} \,. \tag{1.1}$$

It was shown in [DJB12] that there is an r-graph $\mathcal{F}$ satisfying

$$R_{r,\mathcal{F}}(t) \geq 2^{\Omega(t^{r-2})} \,. \tag{1.2}$$

An alternative proof of (1.2) follows from Proposition 3.5.2.

The famous stepping-up lemma of Erdős and Hajnal (see [GRS91]) allows one to transform a construction of an r-graph without a large monochromatic set into an exponentially larger $(r+1)$-graph without a large monochromatic set (of roughly the same size). Observe that both (1.1) and (1.2) suggest that if 2^{t^α} is the size of the largest non-universal r-graph $\mathcal{G}$, so that neither $\mathcal{G}$ nor $\overline{\mathcal{G}}$ contain $K^{(r)}_{t,\ldots,t}$, then the corresponding bound for $(r+1)$-graphs is $2^{t^{\alpha+1}}$. The following theorem establishes one side of this relation, by proving an Erdős–Hajnal-type stepping-up lemma for the problem of bounding $R_{r,\mathcal{F}}(t)$.

Theorem 1.3.3. *The following holds for every $r \geq 4$. For every $(r-1)$-graph $\mathcal{F}$ there is an r-graph $\mathcal{F}^+$ and a constant $c = c(r, \mathcal{F}) > 0$, so that*

$$R_{r,\mathcal{F}^+}(t) \geq \left(R_{r-1,\mathcal{F}}(ct)\right)^{ct} \,.$$

4 While the proof in [VS12] obtained the bound $R_{3,\mathcal{F}}(t) \leq 2^{t^2/f(t)}$ with $f(t)$ an inverse Ackermann-type function (on account of using the hypergraph regularity lemma), the proof in [DJB12] gave the improved bound $R_{3,\mathcal{F}}(t) \leq 2^{t^{2-c}}$ where $c = c(\mathcal{F})$ is a constant that depends only on $\mathcal{F}$.

Theorem 1.3.3 implies that any improvement of (1.2) for $r = 3$ immediately implies a similar improvement of (1.2) for arbitrary $r \geq 3$. In particular, as a corollary of Theorems 1.3.2 and 1.3.3 we obtain the following improvement of (1.2).

Corollary 1.3.4. *For every $r \geq 3$ there is an r-graph $\mathcal{F}$ satisfying $R_{r,\mathcal{F}}(t) \geq t^{\Omega(t^{r-2})}$.*

To prove Theorem 1.3.3 we need to overcome two hurdles. First, we need a way to construct the r-graph $\mathcal{F}^+$ given the $(r-1)$-graph $\mathcal{F}$. An important tool for this step will be an application of a theorem of Alon, Pach and Solymosi [NJJ01], which is a hypergraph extension of a result of Rödl and Winkler [VP89]. The second hurdle is how to construct an r-graph avoiding a $K^{(r)}_{t,\dots,t}$ given an $(r-1)$-graph avoiding a large $K^{(r-1)}_{t',\dots,t'}$. Here we will apply a version of a very elegant argument from [DJB10], which is a variant of the Erdős–Hajnal stepping-up lemma. While this variant of the stepping-up lemma is not as efficient as the original one[5], it is strong enough for our purposes.

A Ramsey variant of the Brown-Erdős-Sós conjecture

The final part of Chapter 3 covers a more recent paper [AM21] (joint with A. Shapira) and concerns with the Brown-Erdős-Sós problem.

The first result in extremal graph theory is probably Mantel's theorem stating that an n vertex graph with more than $n^2/4$ edges contains 3 edges spanned by 3 vertices, that is, a triangle. This is of course just a special case of Turán's theorem, one of the fundamental theorems in graph theory. Turán's theorem spurred an entire branch within graph theory of what is now called Turán-type problems in graphs and hypergraphs [Kee11], as well as in other settings such as matrices and ordered graphs, see [G T18].

One of the most notorious Turán-type problems is a conjecture raised in the early 70's by Brown, Erdős and Sós [BES73b, BES73a]. To state it we need a few definitions. An r-uniform hypergraph (r-graph for short) $\mathcal{G} = (V, E)$ is composed of a vertex set V and an edge set E where every edge in E contains precisely r distinct vertices. An r-graph is *linear* if every pair of vertices belong to at most one edge. We call a set of k edges spanned by at most v vertices a (v, k)-configuration. Then the Brown–Erdős–Sós conjecture (BESC for short) states that for every $k, r \geq 3$ and $\delta > 0$ if $n \geq n_0(k, r, \delta)$ then every linear r-graph on n vertices with at least δn^2 edges contains an $((r-2)k+3, k)$-configuration.

The simplest case of the BESC is when $r = k = 3$. This special case was famously solved by Ruzsa and Szemerédi [RS78] and became

5 Observe that stepping-up lemmas with an exponential blowup-up are *not* useful in our setting since (1.1) and (1.2) tell us that the gap between $R_{r-1,\mathcal{F}}(t)$ and $R_{r,\mathcal{F}^+}(t)$ is *not* exponential.

known as the $(6,3)$-theorem. To get a perspective on the importance of this theorem suffice it to say that the famous *triangle removal lemma* (see [DJ13] for a survey) was devised in order to prove the $(6,3)$-theorem, that one of the first applications of Szemerédi's regularity lemma [E S78] was in [RS78], and that the $(6,3)$-theorem implies Roth's theorem [Rot53] on 3-term arithmetic progressions in dense sets of integers. Despite much effort the problem is wide open already for the next configuration, namely $(7,4)$. As an indication of the difficulty of this case let us mention that it implies the notoriously difficult Szemerédi theorem [Sze75] for 4-term arithmetic progressions (see [P E75]). Let us conclude this discussion by mentioning that the best result towards the BESC was obtained 15 years ago by Sárközy and Selkow [SS04] who proved that $f_3(n, k+2+\lfloor \log_2 k \rfloor, k) = o(n^2)$.[6] Since then, the only advancement was obtained by Solymosi and Solymosi [DJ17] who improved the $f_3(n,15,10) = o(n^2)$ bound of [SS04] to $f_3(n,14,10) = o(n^2)$. Conlon, Gishboliner, Levanzov and Shapira [D C+] have recently announced an improvement of the result of [SS04] that replaces the $\log k$ term with $\log k / \log\log k$.

Given the difficulty of the BESC, researchers have recently looked at various relaxations of it. For example, instead of looking at arbitrary r-graphs, one can look at those arising from a group, see [J L20, RBM20, Sol15, JC20, C W20]. We will consider in this chapter another relaxation of the BESC which was recently suggested independently by Conlon and Nenadov (private communications). We say that a linear r-graph in *complete*[7] if every pair of vertices belong to exactly one edge.

Problem 1.3.5 (Conlon, Nenadov)**.** *Prove that the following holds for every $r \geq 3$, $k \geq 3$, $c \geq 2$ and large enough $n \geq n_0(c,r,k)$: If $\mathcal{G}$ is an n-vertex complete linear r-graph then in every c-colouring of its edges one can find k edges of the same colour, which are spanned by at most $(r-2)k+3$ vertices.*

As we mentioned above, the BESC is a Turán-type question, stating that enough edges force the appearance of certain configurations. With this perspective in mind, Problem 1.3.5 is its natural Ramsey weakening. Indeed, BESC implies its statement, as it gives the required monochromatic configuration in the most popular colour. The relation is analogous to the one between Szemerédi's theorem [Sze75] and Van der Waerden's theorem [Wae27]. In order to get a better feeling of this problem, we encourage the reader to convince themself of the folklore observation that Problem 1.3.5 holds for $c = 1$. A simple application of Ramsey's theorem also shows that Problem 1.3.5 holds when $k = 3$.

6 Here, $f_3(n,v,k)$ is the corresponding extremal number, i.e. the smallest m such that every 3-graph with n vertices and m edges contains a (v,k)-configuration.

7 Such an object is sometimes called an r-Steiner System (when $r = 3$ this is a *Steiner Triple System*). Note that there are many non-isomorphic complete linear r-graphs on n vertices.

Our aim here is to give a positive answer to Problem 1.3.5 assuming r is large enough. More precisely, we have the following.

Theorem 1.3.6. *For every integer c there exists $r_0 = r_0(c)$ such that for every $r \geq r_0$ and integer $k \geq 3$ there exists $n_0 = n_0(c, r, k)$ such that every c-colouring of a complete linear r-graph on $n > n_0$ vertices contains a monochromatic $((r-2)k+3, k)$-configuration.*

Note that even under assumptions of large uniformity it is unlikely that $((r-2)k+3, k)$ can ever be improved to $((r-2)k+2, k)$. Indeed, a conjecture by Füredi and Ruszinkó [ZR13] states that for each $r \geq 3$ there exist arbitrarily large r-Steiner Systems without an $((r-2)k+2, k)$-configuration. That would preclude an extension of Theorem 1.3.6 to $((r-2)k+2, k)$ even for $c = 1$. The case $r = 3$ of the Füredi-Ruszinkó conjecture is an old conjecture by Erdős [P E76], which was recently proved asymptotically, independently in [TL19] and [S G+20].

In the important special case of $c = 2$ we show that $r_0(2)$ can be chosen as small as 4.

Theorem 1.3.7. *For any integers $r \geq 4$ and $k \geq 3$ there exists $n_0 = n_0(r, k)$ such that every 2-colouring of a complete linear r-graph on $n > n_0$ vertices contains a monochromatic $((r-2)k+3, k)$-configuration.*

While we believe that with some effort it should be possible to show that $r_0(2) = 3$, it appears that completely removing the assumption that r is large enough as a function of c would require a different approach. In particular, while the case $k = 3$ is an easy application of Ramsey's theorem, we do not know how to resolve Problem 1.3.5 already for $(c, r, k) = (3, 3, 4)$.

The proof of Theorems 1.3.6 and 1.3.7 has two key ideas. The first is to work with an auxiliary graph B of "bowties". Every vertex v in this graph corresponds to a pair of intersecting[8] edges of the r-graph $\mathcal{G}$. The graph B contains edges only between a vertex b_1, representing two intersecting edges $\{S_1, T\}$ of $\mathcal{G}$ and another vertex b_2, representing two intersecting edges $\{S_2, T\}$ and only if the edges S_1, S_2, T form a $(3r-3, r)$-configuration. In Section 3.7 we will collect several preliminary observations regarding the graph B and about edge-colourings of complete graphs. In Section 3.8 we will prove our main results assuming B has certain nice properties. This will reduce the proof to proving Lemma 3.8.6 which is the main technical part of the proof and is proved in Section 3.9. The second main idea of this proof is to define a somewhat subtle induction which will be used in order to gradually "grow" $((r-2)k+3, k)$-configurations, for $k = 3, 4, \ldots$, and thus prove Lemma 3.8.6. See Section 3.9 for an overview of this proof.

Perhaps one take-home message of this result is that even when considering the Ramsey relaxation of the BESC (stated in Problem

8 Since we only consider linear r-graphs, if two edges intersect, they intersect at a single vertex. We will frequently use this fact throughout.

1.3.5), and even after adding the assumption that $r \geq r_0(c)$, one still has to work quite hard in order to find the $((r-2)k+3, k)$-configurations of the BESC.

1.4 INDUCIBILITY AND EDGE-STATISTICS

Chapter 4 is based on two papers [DM18] (joint work with D. Hefetz) and [NM20] (with N. Alon, D. Hefetz and M. Krivelevich)

A common theme in modern extremal combinatorics is the study of densities or induced densities of fixed objects (such as graphs, digraphs, hypergraphs, etc.) in large objects of the same type, possibly under certain restrictions. This general framework includes Turán densities of graphs and hypergraphs, local profiles of graphs and their relation to quasi-randomness, and more. One such line of research was initiated in 1970s by Pippenger and Golumbic [PG75]. Given graphs G and H, let $D_H(G)$ denote the number of induced subgraphs of G that are isomorphic to H and let $I_H(n) = \max\{D_H(G) : |G| = n\}$. A standard averaging argument was used in [PG75] to show that the sequence $\{I_H(n)/\binom{n}{|H|}\}_{n=|H|}^{\infty}$ is monotone decreasing, and thus converges to a limit $ind(H)$, the so-called *inducibility* of H.

Since it was first introduced in 1975, inducibility has been studied in many subsequent papers. Determining this invariant seems to be a very hard problem. To illustrate the current state of knowledge (or lack thereof), it is worthwhile to note that even the inducibility of paths of length at least 3 and cycles of length at least 6 are not known. Still, the inducibility of a handful of graphs and graph classes is known. These include various very small graphs (see, e.g., [J B+16, EL15, Hir14]) and complete multipartite graphs (see, e.g., [B B+95, BCS86, JA94]). Additional recent results on inducibility can be found, e.g., in [Hua14, MS17]. Some of the recent progress in this area is due to Razborov's theory of flag algebras [Raz07], which provides a framework for systematic computer-aided study of questions of this type.

While, trivially, the complete graph $H = K_k$ and its complement achieve the maximal possible inducibility of 1, the natural analogous question, which graphs on k vertices *minimise* the quantity $ind(H)$, which has been asked in [PG75], is still open.

Let H be an arbitrary graph on k vertices, where k is viewed as large but fixed. By considering a balanced blow-up of H (and ignoring divisibility issues), it is easy to see that $ind(H) \geq k!/k^k$. An iterated blow-up construction provides only a marginally better lower bound of $k!/(k^k - k)$. Pippenger and Golumbic [PG75] conjectured that the latter is tight for cycles.

Conjecture 1.4.1 ([PG75]). *$ind(C_k) = k!/(k^k - k)$ for every $k \geq 5$.*

Note that the requirement $k \geq 5$ appearing in Conjecture 1.4.1 is necessary. Indeed, $ind(C_3) = 1$ since $C_3 = K_3$ is a complete graph and, as shown in [PG75], $ind(C_4) = 3/8$ since $C_4 = K_{2,2}$ is a balanced complete bipartite graph. The authors of [PG75] also posed the following asymptotic version of the above conjecture.

Conjecture 1.4.2 ([PG75])**.** *$ind(C_k) = (1 + o(1))k!/k^k$.*

In support of Conjecture 1.4.2, it was shown in [PG75] that $I_{C_k}(n) \leq \frac{2n}{k}\left(\frac{n-1}{k-1}\right)^{k-1}$ holds for every $k \geq 4$. This implies that $ind(C_k) \leq 2e \cdot k!/k^k$, leaving a multiplicative gap of $2e$ (which is approximately 5.4366) between the known upper and lower bounds. In this chapter we partially bridge the above gap by proving a better upper bound on the inducibility of C_k, namely $ind(C_k) \leq (128/81)e \cdot k!/k^k$ (note that $(128/81)e$ is approximately 4.2955).

Theorem 1.4.3. *For every $k \geq 6$ we have*

$$ind(C_k) \leq \frac{128e}{81} \cdot \frac{k!}{k^k}.$$

We note that the case $k = 5$ of Conjecture 1.4.1 was settled by Balogh, Hu, Lidický and Pfender [J B+16], who showed, in particular, that if n is a power of 5, then $I_{C_5}(n)$ is uniquely attained by the iterated blow-up of C_5. The proof which was given in [J B+16] combines flag algebras [Raz07] and stability methods. It is also worth noting that, in *triangle-free* graphs, all pentagons are induced. Maximising the number of pentagons in triangle-free graphs is an old problem of Erdős [Erd84], which was solved, using flag algebras, by Grzesik [A G12] and independently by Hatami, Hladký, Král', Norine and Razborov [Hat+13] (prior to the use of flag algebras, the best result was due to Győri [Győ89] who gave an elegant elementary proof of a slightly weaker bound).

The second half of Chapter 4 deals with the more general problem of edge-statistics in (large) graphs, a topic which we introduced in [NM20]. Let k be a positive integer and let G be a finite graph of order at least k. Let $A = A_{G,k}$ be chosen uniformly at random from all subsets of $V(G)$ of size k and let $X_{G,k} = e(G[A])$. That is, $X_{G,k}$ is the random variable counting the number of edges of G with both endpoints in A. Naturally, the above quantities can also be interpreted as densities rather than probabilities, and we shall frequently switch between these two perspectives.

Given integers $n \geq k$ and $0 \leq \ell \leq \binom{k}{2}$, let $I(n,k,\ell) = \max\{\mathbb{P}(X_{G,k} = \ell) : |G| = n\}$, that is, $I(n,k,\ell)$ is the maximum density of induced subgraphs with k vertices and ℓ edges, taken over all graphs of order n. A standard averaging argument shows that $I(n,k,\ell)$ is a monotone decreasing function of n. Consequently, we define $ind(k,\ell) := \lim_{n\to\infty} I(n,k,\ell)$ to be the *edge-inducibility* of k and ℓ. While this quantity is trivially 1 for $\ell \in \left\{0, \binom{k}{2}\right\}$ (simply take G to be a large empty or complete graph, respectively), it is natural to ask how large can $ind(k, \ell)$ be for $0 < \ell < \binom{k}{2}$.

This question is closely related to the problem of determining the inducibilities of fixed graphs, a concept which was introduced in 1975 by Pippenger and Golumbic [PG75]. For a graph H, let $D_H(G)$ denote the number of induced subgraphs of G that are isomorphic to H, and let $I_H(n) = \max\{D_H(G) : |G| = n\}$. Again, the sequence $\{I_H(n)/\binom{n}{|H|}\}_{n=|H|}^{\infty}$ is monotone decreasing and thus converges to a limit $ind(H)$, the *inducibility* of H. There has been a lot of interest in this area (see, e.g., [J B+16, DM18, Yus19, KNV19]), in particular, in the first half of this chapter we covered the results on Hefetz and Tyomkyn [DM18] on the inducibility of cycles.

Observe that both types of inducibility are invariant under taking complements, that is, $ind(k, \ell) = ind\left(k, \binom{k}{2} - \ell\right)$ and $ind(H) = ind(\overline{H})$. Note also that $ind(H) \leq ind(|H|, e(H))$. Moreover, if $|H| = k$ and $e(H) \in \left\{1, \binom{k}{2} - 1\right\}$, then $ind(H) = ind(k, e(H))$, as H is the unique (up to isomorphism) graph with k vertices and $e(H)$ edges.

Consider a random graph $G \sim G(n, p)$, where $p = \binom{k}{2}^{-1}$. A straightforward calculation shows that the expected value of the number of k-subsets of $V(G)$ which span precisely one edge is about $1/e$. This implies that $ind(k, 1) \geq 1/e + o_k(1)$ (as the $o_k(1)$ notation suggests, we will often think of k as an asymptotic quantity and, in particular, we will assume k to be sufficiently large wherever needed). In fact, we will see later that there are many constructions which achieve $1/e + o_k(1)$ as a lower bound for $ind(k, 1)$. Another construction, achieving the same asymptotic value for $\ell = k - 1$ is the complete bipartite graph with the smaller part of size n/k, so that $ind(k, k-1) \geq ind(K_{1,k-1}) \geq 1/e + o_k(1)$. In fact, it is known [JA94] that $ind(K_{1,k-1}) = 1/e + o_k(1)$. Note that the $o_k(1)$ term is necessary. For example, counting cherries in $K_{n/2,n/2}$ shows that $ind(3, 2) = ind(K_{1,2}) \geq 3/4$ (in fact, it follows from Goodman's Theorem that $ind(3, 1) = ind(3, 2) = 3/4$). Motivated by the aforementioned constructions (as well as some additional data), we conjecture that the lower bound of $1/e$ is asymptotically tight.

Conjecture 1.4.4 (The Edge-statistics Conjecture)**.** *For every $\varepsilon > 0$ there exists $k_0 = k_0(\varepsilon)$ such that for all integers $k > k_0$ and $0 < \ell < \binom{k}{2}$ we have $ind(k, \ell) \leq 1/e + \varepsilon$.*

For graph-inducibilities we make an analogous conjecture, which would be implied by the Edge-statistics Conjecture.

Conjecture 1.4.5 (The Large Inducibility Conjecture)**.**

$$\limsup \left\{ind(H) : H \notin \{K_{|H|}, \overline{K}_{|H|}\}\right\} = 1/e.$$

Our first theorem on this topic constitutes a first step towards proving Conjecture 1.4.4. It asserts that $ind(k, \ell)$ is bounded away from 1 by an absolute constant for every k and $0 < \ell < \binom{k}{2}$.

Theorem 1.4.6. *There exists an $\varepsilon > 0$ such that for all positive integers k and ℓ which satisfy $0 < \ell < \binom{k}{2}$ we have*

$$ind(k, \ell) < 1 - \varepsilon.$$

For clarity of presentation, we do not give explicit bounds on ε and refer to Section 4.12 for a discussion.

Note that it is not hard to prove that for every positive integer k we have $ind(k, \ell) = 1$ if and only if $\ell \in \left\{0, \binom{k}{2}\right\}$. Indeed, if $0 < \ell < \binom{k}{2}$, then $ind(k, \ell) < 1 - 4^{-k^2}$ is an easy consequence of Ramsey's Theorem and the aforementioned monotonicity of $I(n, k, \ell)$. With a bit more effort, this bound can be improved to $1 - k^{-2}$. On the other hand, we do not see a simple argument that would upper bound $ind(k, \ell)$ away from 1 by an absolute constant as in Theorem 1.4.6. Note also that the related problem of *minimizing* graph-inducibilities has been extensively studied. In particular, Pippenger and Golumbic [PG75] showed that the inducibility of any k-vertex graph is at least $(1 + o_k(1))k!/k^k$. It follows that $ind(H) > 0$ for every graph H and thus $ind(k, \ell) > 0$ for every k and ℓ. We refer the reader to Section 4.12 for further discussion.

For various ranges of values of ℓ (viewed as a function of k) we establish much better upper bounds than the one stated in Theorem 1.4.6. First, for every ℓ satisfying $\min\left\{\ell, \binom{k}{2} - \ell\right\} = \omega(k)$, we prove an upper bound of $1/2$.

Proposition 1.4.7. *For every $\varepsilon > 0$ there exist $C(\varepsilon) > 0$ and $k_0(\varepsilon) > 0$ such that the following holds. Let k and ℓ be integers satisfying $k \geq k_0$ and $Ck \leq \ell \leq \binom{k}{2} - Ck$. Then*

$$ind(k, \ell) \leq \frac{1}{2} + \varepsilon.$$

Next, we prove Conjecture 1.4.4 'almost everywhere'. In fact, we prove a much stronger statement, namely that for every ℓ satisfying $\min\left\{\ell, \binom{k}{2} - \ell\right\} = \Omega\left(k^2\right)$ the quantity $ind(k, \ell)$ is actually polynomially small in k – the right asymptotic behaviour as can be seen by considering the random graph $G(n, \ell/\binom{k}{2})$, which gives $ind(k, \ell) = \Omega(k^{-1})$.

Theorem 1.4.8. *For every positive integers k and ℓ such that* $\min\{\ell, k^2/2 - \ell\} = \Omega(k^2)$ *we have $ind(k, \ell) = O\left(k^{-0.1}\right)$.*

Lastly, we consider the case when ℓ is fixed (i.e., does not depend on k). Here we prove an upper bound of $3/4$. In the interesting sub-case $\ell = 1$, which corresponds to the inducibility of the one-edge graph (equivalently, of K_k^-, the complete graph with one edge removed) we prove a yet better bound of $1/2$.

Theorem 1.4.9. *For every fixed positive integer ℓ we have*

$$ind(k, \ell) \leq \frac{3}{4} + o_k(1).$$

Moreover, for $\ell = 1$ we have

$$ind(k, 1) \leq \frac{1}{2} + o_k(1).$$

Our results are summarized in the following table. For various ranges of $\ell \leq k^2/4$, it states the best known upper bound on $ind(k, \ell)$. Note that for $\ell \geq k^2/4$ the table can be extended symmetrically.

$\ell = \ell(k)$	1	const.	$[\omega(1), O(k)]$	$[\omega(k), o(k^2)]$	$[\Omega(k^2), k^2/4]$
$ind(k, \ell) \leq$	$1/2$	$3/4$	$1 - \varepsilon$	$1/2$	$O(k^{-0.1})$

1.5 TRIANGLES AND ANTI-TRIANGLES

Chapter 5 combines two results (based on papers [DM16] and [M T21]) dealing with the interplay between triangles and ant-triangles (empty 3-vertex sets) in graphs.

A graph is called ℓ-universal if it contains every ℓ-vertex graph as an induced subgraph. Universality is a well-studied graph property, for instance, the famous Erdős-Hajnal conjecture [PA89] can be formulated in the following form.

Conjecture 1.5.1 (Erdős-Hajnal). *For every integer ℓ there exists an $\epsilon > 0$ such that every n-vertex graph G with no clique or independent set of size n^ϵ is ℓ-universal.*

In 2015 Linial and Morgenstern [LM15] asked a question of a similar flavour. Instead of forbidding large cliques and independent sets (anti-cliques) they asked, what happens if the graph G contains only *few* cliques and anticliques of a certain order m. Here we address this question.

First, let us introduce some useful notation and terminology, most of which is standard (see e.g. [Bol98]). For a graph G write $V(G)$ and $E(G)$ for its sets of vertices and edges, respectively. Let $|G| = |V(G)|$ denote the *order* of G and let $e(G) = |E(G)|$ denote its *size*. The *complement* of G is denoted by $\overline{G}$. For a set $S \subseteq V(G)$ put $G[S]$ for the subgraph of G induced on the set S. For a set $S \subseteq V(G)$ and a vertex $u \in V(G)$, let $N_G(u, S) = \{w \in S : uw \in E(G)\}$ denote the set of neighbours of u in S and let $d_G(u, S) = |N_G(u, S)|$ denote the *degree* of u into S. We abbreviate $N_G(u, V(G))$ to $N_G(u)$ and $d_G(u, V(G))$ to $d_G(u)$. The former is referred to as the *neighbourhood* of u in G and the latter as its *degree*. We use $d_G(u, v)$ to denote the *co-degree* of u and v, that is, $|N_G(u) \cap N_G(v)|$ and the somewhat less standard $d_G(u, -v)$ to denote $|N_G(u) \setminus N_G(v)|$. Often, when there is no risk of confusion, we omit the subscript G from the notation above.

For graphs G and H, put $D_H(G)$ for the number of induced copies of H in G and $p_H(G)$ for the corresponding density:

$$p_H(G) = \binom{n}{|H|}^{-1} \cdot D_H(G).$$

The quantity $p_H(G)$ can be also interpreted as the probability that a randomly picked set of $|H|$ vertices of G induces a copy of H.

For $H = K_2$, a single edge, $D_H(G)$ is simply $e(G)$ and thus we write $p_e(G)$ for $p_{K_2}(G)$, the *edge density* of G. For graphs of order 3, since they are determined up to isomorphism by their size, we write $D_i(G)$ for $D_H(G)$ and $p_i(G)$ for $p_H(G)$, where $i = e(H)$. The vector $(p_0(G), \dots, p_3(G))$ is called the 3-*local profile* of G.

Let $\mathcal{G} = (G_k)_{k=1}^{\infty}$ be a sequence of graphs, where $G_k = (V_k, E_k)$ is of order $n_k := |V_k|$ and n_k tends to infinity with k. If for some graph parameter λ the limit $\lim_{k\to\infty} \lambda(G_k)$ exists, we denote it by $\lambda(\mathcal{G})$. A sequence $\mathcal{G}$ is said to be ℓ-*universal* if G_k is ℓ-universal for every sufficiently large k.

Linial and Morgenstern proved in [LM15] that there exists a constant $\rho = 0.159181\dots$ such that every $\mathcal{G}$ with $p_0(\mathcal{G}), p_3(\mathcal{G}) < \rho$ is 3-universal and asked whether an analogous result holds for higher universalities.

Question 1.5.1 ([LM15]). *Given $\ell \geq 4$, is there some $\epsilon > 0$ such that every graph sequence $\mathcal{G}$ with $p_0(\mathcal{G}), p_3(\mathcal{G}) < \frac{1}{8} + \epsilon$ is ℓ-universal?*

Note that our definition of ℓ-*universal* sequences is slightly different from the one given in [LM15]. The latter required additionally that $p_{G_k}(H)$ be bounded away from 0 for each H of order ℓ. However for our purposes (i.e. answering Question 1.5.1) these definitions are equivalent due to the induced graph removal lemma of Alon, Fischer, Krivelevich and Szegedy [Alo+00].

It was pointed out by the author that for every $\ell \geq 5$ the answer to Question 1.5.1 is negative. Though his counterexample has already appeared in [LM15], for the sake of completeness we will repeat it in the next section.

This leaves $\ell = 4$ as the only remaining open case of Question 1.5.1. Our first main result in this chapter, Theorem 1.5.2, answers it in the affirmative, thereby settling Question 1.5.1 in full.

Let us define a sequence of graphs $\mathcal{G}$ to be *t-random-like*, or *tRL* for brevity, if $p_{K_t}(\mathcal{G}) = p_{\overline{K_t}}(\mathcal{G}) = 2^{-\binom{t}{2}}$. Our choice of terminology stems from the fact that such a sequence has approximately the same number of t-cliques and t-anticliques, that is, independent sets of size t, as the random graph $\mathcal{G}_{n,1/2}$. Note that for $\mathcal{G}$ to be 2RL it is sufficient to have $p_e(\mathcal{G}) = 1/2$. We will be mostly interested in 3RL sequences; in our terminology $\mathcal{G}$ is 3RL if and only if $p_0(\mathcal{G}) = p_3(\mathcal{G}) = 1/8$.

A standard diagonalisation argument shows that in order to answer Question 1.5.1 for $\ell = 4$ affirmatively, it suffices to prove the following assertion.

Theorem 1.5.2. *Every 3RL sequence is 4-universal.*

Theorem 1.5.2 is related to the *quasirandomness* of graphs as well. This is a central notion in extremal and probabilistic graph theory. It was introduced by Thomason in [A T87] and was extensively studied in many subsequent papers. In particular, it was proved by Chung, Graham and Wilson [CGW89] (see also [AS04] for more details) that if $p_H(\mathcal{G}) = p_H(\mathcal{G}_{n,1/2})$ holds for every graph H of order 4, then the same equality holds for every graph H of *any* fixed size. In the terminology of [CGW89] this fact is denoted by $P_1(4) \Rightarrow P_1(s)$. On the other hand, it was pointed out in [CGW89] that the property $P_1(3)$, that is, containing the "correct" number of induced copies of every 3-vertex graph, is not sufficient to ensure quasirandomness. As we shall see in Section 5.2, $P_1(3)$ is in fact equivalent to 3RL. Thus, our results can be viewed as the study of $P_1(3)$. Under this viewpoint Theorem 1.5.2 shows that, while 3RL graphs need not satisfy $P_1(4)$, they still must contain a positive density of every possible induced 4-vertex graph.

Having resolved Question 1.5.1, we know that the 3RL property implies 4-universality, but is not enough to ensure ℓ-universality for any larger ℓ. A natural follow up question to ask is, whether there still exist infinite classes of graphs H that must be contained in every 3RL sequence $\mathcal{G}$. Cliques, paths, cycles and stars are natural candidates for such classes. We shall answer this question in the negative by providing counterexamples for each of these classes. In fact, our second main result, Theorem 1.5.3 provides, perhaps surprisingly, a counterexample for *any single* graph which is not too small. Throughout this chapter $R(k, \ell)$ will stand, as usual, for the corresponding Ramsey number (see [Bol98] for more background details).

Theorem 1.5.3. *For every graph H of order at least $R(10, 10)$ there exists a 3RL sequence $\mathcal{G}$, where no $G_k \in \mathcal{G}$ contains a copy of H as an induced subgraph.*

According to [S R11], the best currently known bounds on $R(10, 10)$ are $798 \le R(10, 10) \le 23556$ (the standard upper bound for Ramsey numbers yields $R(10, 10) \le \binom{10+10-2}{10-1} = 48620$).

Theorem 1.5.3 combined with Theorem 1.5.2 and the induced graph removal lemma [Alo+00] immediately give the following corollary.

Corollary 1.5.2. *There exists an $\epsilon > 0$ such that for every graph H of order at least $R(10, 10)$ there is a sequence $\mathcal{G}$, where no $G_k \in \mathcal{G}$ contains an induced copy of H, but $p_J(\mathcal{G}) > \epsilon$ for every 4-vertex graph J.*

Theorem 1.5.3 and Corollary 1.5.2 show that, for sufficiently large values of ℓ, having either the "correct" densities of triangles and anti-triangles or positive densities of every 4-vertex graph is far from being enough to ensure ℓ-universality. This goes in stark contrast with $\mathcal{G}$ having the "correct" densities of *all* induced 4-vertex graphs, which implies that $\mathcal{G}$ is quasirandom and therefore ℓ-universal for every ℓ.

Having constructions of 3RL sequences which are only ℓ-universal for very small values of ℓ on the one hand and the random graph

$\mathcal{G}_{n,1/2}$ (which is ℓ-universal for every fixed ℓ) on the other hand, it is natural to ask, if for arbitrarily large ℓ there exists a 3RL sequence which is ℓ-universal but not $f(\ell)$-universal for some function f. This would show that no fixed universality is sufficient to ensure all other universalities. Our third theorem shows that this is indeed the case in the following strong sense.

Theorem 1.5.4. *For every ℓ there exists a 3RL sequence $\mathcal{G}_\ell$ such that $p_H(\mathcal{G}_\ell) > 0$ for every graph H of order 2^ℓ, but $\mathcal{G}_\ell$ is not $24\ell \cdot 2^\ell$-universal.*

In the second part of Chapter 5 we cover the content of the recent solo paper [M T21]. One of the classical results in extremal graph theory, Goodman's theorem [Goo59], states that in every 2-colouring of the edges of the complete graph K_n the number of monochromatic triangles is at least $\frac{1}{4}\binom{n}{3} - o(n^3)$. In other words, about a quarter of all possible triangles are guaranteed to be monochromatic. With this in mind, Erdős [P E97, Erd+97] asked about the number of *edge-disjoint* monochromatic triangles in any 2-colouring of K_n.

To be more formal, a *triangle packing* of a graph G is a collection of edge-disjoint triangles in G. The *size* of a triangle-packing is the total number of edges it contains.[9] Define $f(n)$ to be the largest number m such that every 2-colouring of the edges of K_n contains a triangle packing of size m in which each triangle is monochromatic.

As a basic example, consider $n = 6$. By the folklore fact about Ramsey numbers, any 2-colouring of K_6 contains a monochromatic triangle, and it is not hard to see that it has to contain at least two such triangles. However, they need not be edge-disjoint, as can be seen by taking a 5-cycle and replicating a vertex. So $f(6) = 3$.

In general, the obvious upper bound of $f(n) \leq n^2/4 - o(n^2)$ is seen to hold by considering the balanced complete bipartite graph and its complement. Erdős [P E97, Erd+97] conjectured that this is tight.[10]

Conjecture 1.5.3.

$$f(n) = \frac{n^2}{4} - o(n^2).$$

To draw a parallel to Goodman's theorem, Conjecture 1.5.3 states that every 2-edge-colouring of K_n admits a packing with monochromatic triangles, containing about one half of all possible edges.

In previous works, Erdős, Faudree, Gould, Jacobson and Lehel [Erd+97] proved a first non-trivial lower bound of $f(n) \geq (9/55)n^2 + o(n^2)$. Keevash and Sudakov [PB04] improved this, by using the fractional relaxation of the problem, to $f(n) \geq n^2/4.3 + o(n^2)$. Alon and

9 This is obviously the number of the triangles in the packing times 3. We prefer the present scaling for technical and presentation reasons.

10 In [P E97, Erd+97, PB04] the $n^2/4 + o(n^2)$ notation is used. It is understood that the additive $o(n^2)$-term can be negative, as this is the case e.g. in the above example. Hence, we believe the expression $n^2/4 - o(n^2)$ better reflects the nature of the conjecture.

Linial (see [PB04]) suggested, as a step towards Conjecture 1.5.3, to consider the natural class of colourings, in which one of the colour classes is triangle-free.

At this stage it will be more convenient to break the symmetry and speak of a graph and its complement. A graph is said to be *co-triangle-free* if its complement is triangle-free. Equivalently, co-triangle-free graphs are graphs with independence number at most 2. Define $g(n)$ to be the largest number m such that every co-triangle-free graph on n vertices contains a triangle packing of size m. The same example as for $f(n)$ – the disjoint union of two cliques of order $n/2$, shows that $g(n) \leq n^2/4 - o(n^2)$, and Conjecture 1.5.3 would imply that this is tight.

Conjecture 1.5.4.

$$g(n) = \frac{n^2}{4} - o(n^2).$$

Yuster [R Y07] worked specifically on Conjecture 1.5.4 and proved that any potential counterexample to it must have between $0.2501n^2$ and $3n^2/8$ edges. That is, its size cannot be too close to, or too far from the Mantel threshold.

Our aim is to give a short proof of Conjecture 1.5.4.

Theorem 1.5.5. *We have*

$$g(n) = \frac{n^2}{4} - o(n^2).$$

Moreover, we classify the extremal graphs. An n-vertex graph is said to be *ε-far* from being bipartite if at least εn^2 edge deletions are required in order to make it bipartite.

Theorem 1.5.6. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that any co-triangle-free graph G of order n, whose complement is ε-far from being bipartite, has a triangle packing of size $(1/4 + \delta)n^2 + o(n^2)$.*

We say that a graph is *co-bipartite* if its complement is bipartite. Equivalently, G is co-bipartite if $V(G)$ is spanned by a disjoint union of two cliques; clearly, co-bipartite graphs are co-triangle-free. Thus, Theorems 1.5.5 and 1.5.6 imply that every co-triangle free graph on n vertices admits a triangle packing on $n^2/4 - o(n^2)$ edges, and the graphs achieving at most $n^2/4 + o(n^2)$ are essentially co-bipartite.

At the core of our proof is Lemma 5.9.2. It deals with the case when G is 'critical', that is its complement $\overline{G}$ is triangle-free and not bipartite, but can be made bipartite by deleting a vertex. Lemma 5.9.2 states that G has a fractional triangle packing of size larger than $n(n-1)/4$. This, combined with the integer-fractional transference principle of Haxell and Rödl, averaging over fractional packings, and a computer verification for small values of n in the spirit of [PB04], yields the proof of Theorem 1.5.5.

To prove Theorem 1.5.6, in addition to the above tools, we apply a theorem of Alon, Shapira and Sudakov on the structure of graphs with a large edit distance to a monotone graph property.

1.6 MONOCHROMATIC CONNECTED COMPONENTS

The results of Chapter 5 can also be interpreted as Ramsey problems for 2-colourings of complete graphs. Continuing this line of thought, in Chapter 6 (following papers [DM23] and [DM22]) we turn our attention to another natural Ramsey-type question in graphs. Namely, given an r-edge-colouring of the complete graph K_n, what is the largest number of edges in a monochromatic connected component? This natural question has only recently received the attention it deserves, partly due to the results presented here.

First, following [DM23] (joint with D. Conlon), we show that the answer for $r = 2$ is $2n^2/9 + o(n^2)$. In fact, we prove a more general statement, namely that every two-colouring of the edges of the complete graph K_n contains a monochromatic trail or circuit of length at least $2n^2/9 + o(n^2)$. This is asymptotically best possible.

Next, based on the recent paper [DM22] (joint with D. Conlon and S. Luo) we introduce a general framework for studying this problem and apply it to fully resolve the $r = 4$ case (the case $r = 3$ was previously resolved by S. Luo). If we write $M(n,r)$ for the largest natural number such that every r-colouring of K_n contains a monochromatic connected component with at least $M(n,r)$ edges, then the main result of [DM23] and Section 6.1 may be interpreted as saying that $M(n,2) = \frac{2}{9}n^2 + o(n^2)$. In fact, a more careful analysis of our argument implies that $M(n,2) \geq \frac{1}{9}(2n^2 - n - 1)$, with, where divisibility allows it, the example consisting of two disjoint red cliques of orders $\frac{2n+1}{3}$ and $\frac{n-1}{3}$ with all blue edges between showing that this is best possible.

To say something about the general case, we first look at Gyárfás' construction of r-colourings where each monochromatic component has at most $n/(r-1)$ vertices. This construction, which relies on the existence of the affine plane of order $r-1$, works when $r-1$ is a prime power and n is a multiple of $(r-1)^2$. Concretely, the affine plane of order $r-1$ corresponds to a copy of $K_{(r-1)^2}$ together with r different decompositions of this graph into $r-1$ vertex-disjoint copies of K_{r-1} (that is, r different K_{r-1}-factors) with the property that any edge is contained in exactly one of the $r(r-1)$ copies of K_{r-1}. By giving the edges in the ith K_{r-1}-factor colour i, we obtain an r-colouring where every monochromatic component has at most $r-1$ vertices. Moreover, when n is a multiple of $(r-1)^2$, we can simply blow up this colouring to obtain an r-colouring where every monochromatic component has at most $n/(r-1)$ vertices.

Essentially the same construction works in the edge case to show that, when $r-1$ is a prime power, there are r-colourings where each monochromatic component has at most $(\frac{1}{r(r-1)}+o(1))\binom{n}{2}$ edges (the only caveat is that we should use each colour roughly the same number of times within each of the blown-up vertices). Luo [Luo], who proved that this is tight for $r=3$. That is, $M(n,3)=\lceil\frac{1}{6}\binom{n}{2}\rceil$ for n sufficiently large. Moreover, by giving a tight lower bound for the largest number of edges in a connected component in a graph of given density, he was able to show that $M(n,r)\geq\frac{1}{r^2}\binom{n}{2}$ in the general case. Our concern here will be with the following conjectural improvement to this bound.

Conjecture 1.6.1. *For any natural numbers n and r with* $r\geq 2$*,*

$$M(n,r)\geq\left\lceil\frac{1}{r(r-1)}\binom{n}{2}\right\rceil.$$

Moreover, when there is no affine plane of order $r-1$*, there exists a constant* $\varepsilon_r>0$ *such that*

$$M(n,r)\geq\left(\frac{1}{r(r-1)}+\varepsilon_r\right)\binom{n}{2}.$$

The results of [DM23] (Section 6.1) and [Luo] prove this conjecture when $r=2$ and 3, respectively. Our main result here is a proof of the next open case, when $r=4$. Note that, in this case, Gyárfás' construction corresponds to a 4-colouring of K_9 where each colour class is the union of three vertex-disjoint triangles. In the statement below, by saying that a colouring matches Gyárfás' construction, we mean that the set of components and the intersection pattern of their vertex sets match those in this construction.

Theorem 1.6.2. *In every 4-colouring of the edges of* K_n*, there is a monochromatic component with at least* $\frac{1}{12}\binom{n}{2}$ *edges. That is,* $M(n,4)\geq\lceil\frac{1}{12}\binom{n}{2}\rceil$*. Moreover, unless the colouring matches Gyárfás's construction, there is a monochromatic component with at least* $\left(\frac{1}{12}+\varepsilon\right)\binom{n}{2}$ *edges, where* $\varepsilon=\frac{2}{14+\sqrt{96}}-\frac{1}{12}>0.0007$*.*

Our proof of Theorem 1.6.2 consists of first showing that any 4-colouring of K_n has one of a bounded number of component structures and then that each such component structure contains a component with enough edges. For instance, one of the possible component structures is that each colour has precisely three components. But then one of these 12 components clearly contains at least $1/12$ of the edges, as required. For the other possible component structures, our arguments are not usually so simple, relying instead on a key observation, Proposition 6.3.1 below. This states that if a certain union of components is large in the vertex sense, but none of these components is large in the edge sense, then some one of the remaining components will be large in the edge sense. In fact, even this is not quite enough and, inspired by Füredi's approach to the vertex case, we must allow for weighted or fractional unions of components.

1.7 REPEATED PATTERNS IN PROPER COLOURINGS

In Chapter 7, based on paper [DM21b] (joint work with D. Conlon) we turn our attention to *proper* edge-colourings of K_n. That is, colourings where any two edges sharing a vertex receive different colours.

A considerable body of recent work in extremal combinatorics is devoted to the study of rainbow patterns in proper edge-colourings of complete graphs. To mention two such results (amongst many [BPS17, BPS20, CP20, EGJ20, Gao+21, GJ20, Glo+21, KY18] [Kim+20, MPS19, MPS20, Pok18, PS18, PS19]), there is the work of Alon, Pokrovskiy and Sudakov [APS17] showing that any proper edge-colouring of K_n contains a rainbow path of length $n - o(n)$ and the work of Montgomery, Pokrovskiy and Sudakov [MPS21] and, independently, Keevash and Staden [PK20] resolving a celebrated conjecture of Ringel, one of whose statements involves finding a rainbow copy of any tree with n edges in a particular proper edge-colouring of K_{2n+1}. For the most part, this recent work has focused on finding large structures in proper edge-colourings. We instead study small structures, our aim being to understand when a proper edge-colouring contains two or more repeats of a particular graph H.

To be more precise, we say that two copies of a graph H in a colouring of K_n are *colour isomorphic* if there exists an isomorphism between them preserving the colours. The following function is our main object of study.

Definition 1. *For $k, n \geq 2$ and a graph H, define $f_k(n, H)$ to be the smallest integer C such that there is a proper edge-colouring of K_n with C colours containing no k vertex-disjoint colour-isomorphic copies (or 'repeats') of H.*

We make several remarks about this definition. First, one could, in principle, ask the same question without the restriction to proper colourings. However, this changes the character of the question completely. Indeed, consider the colouring of the complete graph on vertex set $\{1, 2, \ldots, n\}$ where we colour the edge ij with $i < j$ by the colour i. Then this is a colouring with $n - 1$ colours which does not even contain two disjoint edges of the same colour. On the other hand, when we restrict to proper colourings, we have that $f_k(n, K_2) \geq \lceil \frac{1}{k-1}\binom{n}{2} \rceil$ by a straightforward application of the pigeonhole principle. For n sufficiently large in terms of k, it also follows from several well-known decomposition results, such as Gustavsson's theorem [Gus91], that this bound is tight.

Our second remark collects together several simple observations that we will use throughout.

Remark 1.7.1. The quantity $f_k(n, H)$ is monotone increasing in n, but decreasing in k and in H (with respect to taking subgraphs). Moreover, since every proper colouring has at least $n - 1$ colours,

$$\binom{n}{2} \geq f_k(n, H) \geq n - 1.$$

Finally, although our definition contains no requirement that the copies of H should be rainbow, all of the results where we find repeats of a particular graph H remain true up to a constant factor if we insist that each copy is rainbow. This then brings our work more fully in line with the body of research discussed at the outset. With these preliminaries out of the way, we now describe our main results.

In the classical Turán problem, the growth rate of the extremal function $\mathrm{ex}(n, H)$ is subject to a well-known trichotomy. Namely, non-bipartite graphs, bipartite graphs with a cycle and forests satisfy $\mathrm{ex}(n, H) = \Theta(n^2)$, $n^{1+\Omega(1)} \leq \mathrm{ex}(n, H) \leq n^{2-\Omega(1)}$ and $\mathrm{ex}(n, H) = \Theta(n)$, respectively. Our first theorem shows that something broadly similar holds for $f_2(n, H)$, although, unlike the extremal function, our function can, and usually does, degenerate for bipartite graphs with a cycle. Note that here and throughout, all terms in the O-notation are to be interpreted with respect to n, with all other variables treated as constants.

Theorem 1.7.2. *The growth rate of $f_2(n, H)$ satisfies:*

(i) $f_2(n, H) = \Theta(n^2)$ *if H is a forest. Otherwise,* $f_2(n, H) = O(n^{2-\Omega(1)})$.

(ii) If H is non-bipartite, then $f_2(n, H) \leq n + 1$.

(iii) If H is bipartite and $e(H) \geq 2|H| - 2$, *then* $f_2(n, H) = \Theta(n)$.

(iv) There exist bipartite graphs H with $n^{1+\Omega(1)} \leq f_2(n, H) \leq n^{2-\Omega(1)}$.

For three or more repeats, the class of graphs for which we know that $f_k(n, H) = O(n)$ grows as k increases. In fact, for any graph H containing a cycle, we can show, by using a variant of Bukh's random algebraic method [Buk15], that there exists k such that $f_k(n, H) = O(n)$.

Theorem 1.7.3. *For any graph H containing a cycle, there exists k such that $f_k(n, H) = O(n)$.*

For trees, the situation is much more involved, as spelled out in the next theorem, whose proof relies on a mixture of novel combinatorial and algebraic methods.

Theorem 1.7.4. *For any tree T with m edges and any $k \geq 3$:*

(i) $f_k(n, T) = \Omega(n^{k/(k-1)})$. *Moreover, if T has at least two edges, then* $f_3(n, T) = \Theta(n^{3/2})$.

(ii) $f_k(n, T) = \Omega(n^{(m+1)/m})$ *and there exists k' such that* $f_{k'}(n, T) = O(n^{(m+1)/m})$.

1.8 WEAK SATURATION IN GRAPHS AND HYPERGRAPHS

Chapter 8 deals with my recent work on weak saturation in graphs and hypergraphs. It covers papers [DM21a] (joint work with D. Bulavka and M. Tancer) and [AM22] (with A. Shapira).

Let F and H be q-uniform hypergraphs (q-graphs for short); we identify hypergraphs with their edge sets. We say that a subgraph $G \subseteq F$ is *weakly H-saturated* in F if the edges of $F \setminus G$ can be ordered as $e_1, \dots, e_k$ such that for all $i \in [k]$ the hypergraph $G \cup \{e_1, \dots, e_i\}$ contains an isomorphic copy of H which in turn contains the edge e_i. We call such $e_1, \dots, e_k$ an *H-saturating sequence* or *saturation process* of G in F. The *weak saturation number* of H in F, $wsat(F, H)$ is the minimum number of edges in a weakly H-saturated subgraph of F. When F is complete of order n, we simply write $wsat(n, H)$.

Weak saturation was introduced by Bollobás [Bol68a] in 1968 and is related to (strong) graph saturation: G is H-saturated in F if adding any edge of $F \setminus G$ would create a new copy of H. However, a number of properties of weak saturation make it a more natural object of study. Firstly, it follows from the definition that any graph G achieving $wsat(F, H)$ has to be H-free (we could otherwise remove an edge from a copy of H in G resulting in a smaller example), while for strong saturation H-freeness may or may not be imposed, resulting in two competing notions (see [MS15] for a discussion). Secondly, a short subadditivity argument originally due to Alon [Alo85] shows that for every 2-uniform H, $\lim_{n\to\infty} wsat(n, H)/n$ exists. Whether the same holds for strong saturation is a longstanding conjecture of Tuza [Tuz86]. And thirdly, weak saturation lends itself to be studied via algebraic methods, thus offering insight into algebraic and matroid structures underlying graphs and hypergraphs.

The most natural case when F and H are cliques was the first to be studied. Let K_r^q denote the complete q-graph of order r. Confirming a conjecture of Bollobás, Lovász [Lov77] proved that $wsat(n, K_r^q) = \binom{n}{q} - \binom{n-r+q}{q}$ and independent proofs were given later by Alon [Alo85], Frankl [Fra82], and Kalai [Kal84b, Kal85]. While the upper bound is a construction that is easy to guess (a common feature in weak saturation problems), all of the above lower bound proofs rely on algebraic or geometric methods, and no purely combinatorial proof is known to this date.

In the subsequent years weak saturation has been studied extensively [Alo85, EFT91, Pik01a, Tuz88, MS15, Pik01b, Sem97, BS02, Sid07, FG14, Bal+12, BP98, MN18]. Despite this, our understanding of weak saturation numbers is still rather limited.

In this chapter we first address the case when $H = K^q_{r_1,\dots,r_d}$ is a complete d-partite q-graph for arbitrary $d \geq q > 1$. That is, $V(H)$ is a disjoint union of sets $R_1, \dots, R_d$ with $|R_i| = r_i$ and

$$E(H) = \left\{ e \in \binom{V(H)}{q} : |e \cap R_i| \leq 1 \text{ for all } i \in [d] \right\},$$

in particular, for $q = 2$ we recover the usual complete multipartite graphs. This is perhaps the next most natural class of hypergraphs to consider after the cliques.

For the host graph F, besides the clique it is natural to consider a larger complete d-partite q-graph $K^q_{n_1,\dots,n_d}$. In the latter case we again have a choice between the *undirected* and *directed* versions of the problem. The former follows the definition of weak saturation given at the beginning, while in the latter we additionally impose that the new copies of H in F created in every step "point the same way", i.e. have r_i vertices in the i-th partition class for all $i \in [d]$ (see below for a formal definition).

All three above versions have been studied in the past. For $q = 2$, Kalai [Kal85] determined $wsat(n, K_{r,r})$. Kronenberg, Martins and Morrison [KMM21] gave recently a new proof of this result, extending it to $wsat(n, K_{r,r-1})$ and asymptotically to all $wsat(n, K_{s,t})$. No other values $wsat(n, K^q_{r_1,\dots,r_d})$ are known except for $r_1 = \cdots = r_d = 1$ when H is a clique and a handful of closely related cases, e.g., when all r_i but one are 1 [Pik01b]. When both H and F are complete d-partite, for $d = q$ Alon [Alo85] solved the problem in the directed setting. Moshkovitz and Shapira [MS15], building on Alon's work, settled the undirected case, determining $wsat(K^d_{n_1,\dots,n_d}, K^d_{r_1,\dots,r_d})$. There has been no progress for $d > q$.

In our first contribution in this chapter we settle completely the directed case for all q and d. To state the problem formally, let $\mathbf{r} = (r_1, \dots, r_d)$ and $\mathbf{n} = (n_1, \dots, n_d)$ be integer vectors such that $1 \leq r_i \leq n_i$. Suppose $N = N_1 \sqcup \cdots \sqcup N_d$ where $|N_i| = n_i$ and $\sqcup$ denotes a disjoint union. Let $K^q_{\mathbf{n}}$ be the complete d-partite q-graph on N whose partition classes are the N_i, and let $K^q_{\mathbf{r}}$ be an unspecified complete d-partite q-graph on the same partition classes, with r_i vertices in each N_i. Given a subgraph G of $K^q_{\mathbf{n}}$, a sequence of missing edges $e_1, \dots, e_k$ is a *(directed)* $K^q_{\mathbf{r}}$*-saturating sequence of* G *in* $K^q_{\mathbf{n}}$ if: (i) $K^q_{\mathbf{n}} \setminus G = \{e_1, \dots, e_k\}$; (ii) for every $j \in [k]$ there exists $H_j \subseteq G \cup \{e_1, \dots, e_j\}$ isomorphic to $K^q_{\mathbf{r}}$ such that $e_j \in H_j$ and $|V(H_j) \cap N_i| = r_i$ for all $i \in [d]$. The q-graph G is said to be *(directed) weakly* $K^q_{\mathbf{r}}$*-saturated in* $K^q_{\mathbf{n}}$ if it admits a $K^q_{\mathbf{r}}$-saturating sequence in the latter. The *(directed) weak saturation number of* $K^q_{\mathbf{r}}$ *in* $K^q_{\mathbf{n}}$, in notation $w(K^q_{\mathbf{n}}, K^q_{\mathbf{r}})$, is the minimal number of edges in a weakly $K^q_{\mathbf{r}}$-saturated subgraph of $K^q_{\mathbf{n}}$.

Theorem 1.8.1. *For all $d \geq q \geq 2$, $\mathbf{n}$ and $\mathbf{r}$ we have*

$$w(K^q_{\mathbf{n}}, K^q_{\mathbf{r}}) = \sum_{I \in \binom{[d]}{q}} \prod_{i \in I} n_i - \sum_{I \in \binom{[d]}{\leq q}} \prod_{i \in I} (n_i - r_i).$$

In the above formula $\binom{[d]}{\leq q}$ stands for the set of all subsets of $[d]$ of size at most q, and we use the convention that $\prod_{i \in \emptyset}(n_i - r_i) = 1$.

As mentioned, the $d = q$ case of Theorem 1.8.1 was proved by Alon [Alo85]. Hence our result generalizes Alon's theorem to arbitrary $d \geq q$. When H is balanced, that is when $r_1 = \cdots = r_d$, there is no difference between the directed and undirected partite settings. Writing $K^q(r;d)$ for $K^q_{r,\ldots,r}$ (d times), Theorem 1.8.1 thus determines the weak saturation number of $K^q(r;d)$ in complete d-partite q-graphs.

Corollary 1.8.2. *For all $d \geq q \geq 2$ and $n_1, \ldots, n_d \geq r \geq 1$ we have*

$$wsat(K^q_{n_1,\ldots,n_d}, K^q(r;d)) = \sum_{I \in \binom{[d]}{q}} \prod_{i \in I} n_i - \sum_{I \in \binom{[d]}{\leq q}} \prod_{i \in I} (n_i - r).$$

Our proof of Theorem 1.8.1 combines exterior algebra techniques in the spirit of [Kal85] with a new ingredient: the use of the coluorful exterior algebra inspired by the recent work of Bulavka, Goodarzi and Tancer on the colourful fractional Helly theorem [BGT21].

Kronenberg, Martins and Morrison ([KMM21], section 5) remarked that while the values $wsat(n, K_{t,t})$ and $wsat(K_{\ell,m}, K_{t,t})$ for $\ell + m = n$, which were determined in separate works, are of the same order of magnitude, it is not obvious if there is any direct connection. In our second contribution in this chapter we establish such a connection using a tensoring trick. As we have mentioned earlier, 2-graphs H satisfy $wsat(n, H) = c_H n + o(n)$, and Alon's proof of this fact [Alo85] can be straightforwardly adjusted to show that $wsat(K_{n,n}, H) = c'_H \cdot 2n + o(n)$ when H is bipartite. We show that in fact $c_H = c'_H$. A minor adjustment to our proof gives that, for any rational $0 < \alpha < 1$, the quantities $wsat(n, H)$ and $wsat(K_{\alpha n,(1-\alpha)n}, H)$, when $\alpha n \in \mathbb{Z}$, are of the same order of magnitude. Setting $H = K_{t,t}$ answers the above question of [KMM21]. For $q \geq 3$ a similar method determines asymptotically the weak saturation number of complete d-partite d-graphs in the clique, generalizing Theorem 4 of [KMM21].

Theorem 1.8.3. *For every bipartite 2-uniform graph H we have*

$$\lim_{n\to\infty} \frac{wsat(n, H)}{n} = \lim_{n\to\infty} \frac{wsat(K_{n,n}, H)}{2n}. \tag{1.3}$$

Furthermore, for any $d \geq 2$ and $1 \leq r_1 \leq \cdots \leq r_d$ we have

$$wsat(n, K^d_{r_1,\ldots,r_d}) = \frac{r_1 - 1}{(d-1)!} n^{d-1} + O(n^{d-2}). \tag{1.4}$$

In the second half of the chapter we study the limiting constant in more generality. Recall that $wsat(n, H)$ stands for the smallest number of edges in a weakly H-saturated r-graph on n vertices.

Note that by the construction from [EHM64], we know that every graph H we have $wsat(n, H) = O_H(n)$. As of now, the best known general bounds for $wsat(n, H)$ when H is a graph are due to Faudree, Gould and Jacobson [FGJ13] who showed that for graphs H of minimum degree $\delta = \delta(H)$ we have[11]

$$\left(\frac{\delta}{2} - \frac{1}{\delta+1}\right) \cdot n \leq wsat(n, H) \leq (\delta - 1) \cdot n + O(1).$$

At this point it is natural to ask if for every H there is a constant C_H so that

$$wsat(n, H) = (C_H + o(1))n. \tag{1.5}$$

Such a result was obtained in 1985 by Alon [Alo85], who proved that for graphs the function $wsat(n, H)$ is (essentially) subadditive, implying that $wsat(n, H)/n$ tends to a limit, by Fekete's subadditivity lemma [Fek23].

Much less was known when H is an r-graph with $r \geq 3$. Similarly to the case $r = 2$ above, Bollobás's construction from [Bol65] gives a simple bound of

$$wsat(n, H) = O_H(n^{r-1}).$$

A more refined result was obtained by Tuza [Tuz92] who introduced the following key definition. The *sparseness* of an r-graph H, denoted $s(H)$, is the smallest size of a vertex set $W \subseteq V$ contained in precisely one edge of H; note that $1 \leq s(H) \leq r$ for every non-empty r-graph H. It was proved in [Tuz92] that for every r-graph H there are two positive reals c_H and C_H such that

$$c_H \cdot n^{s-1} \leq wsat(n, H) \leq C_H \cdot n^{s-1}. \tag{1.6}$$

It was further conjectured in [Tuz92] that the more refined bound $wsat(n, H) = C_H \cdot n^{s-1} + O(n^{s-2})$ holds for every r-graph of sparseness s. See also the recent survey [Cur+21] on saturation problems where this conjecture is further discussed. Since such a result is not known even for graphs (i.e. when $r = s = 2$), Tuza [Tuz92] asked if one can improve upon (1.6) by showing that for every r-graph we have $wsat(n, H) = C_H \cdot n^{s-1} + o(n^{s-1})$ where $s = s(H)$. Prior to this work, such a result was only known for $r = 2$ by Alon's result (1.5). Here we fully resolve Tuza's problem for all r-graphs.

Theorem 1.8.4. *For every r-graph H there is $C_H > 0$ such that*

$$\lim_{n\to\infty} wsat(n, H)/n^{s-1} = C_H,$$

where $s = s(H)$ is the sparseness of H. In particular[12]*, for every r-graph H there is $C'_H \geq 0$ such that*

$$\lim_{n\to\infty} wsat(n, H)/n^{r-1} = C'_H.$$

11 The upper bound is known to be tight for many graphs, the cliques being one example. Concerning the lower bound, the authors of [FGJ13] give a construction of a graph H with $wsat(n, H) \leq (\delta/2 + 1/2 - 1/\delta)n$.

12 Here we simply use the fact that for every r-graph H we have $1 \leq s(H) \leq r$.

It is interesting to note that Tuza [Tuz86] (for graphs) and Pikhurko [Pik99] (for arbitrary r-graphs) also conjectured that a theorem analogous to the second assertion of Theorem 1.8.4 should hold with respect to $sat(n, H)$. However, there are results suggesting that this analogous statement does not hold even for graphs, see [Beh18, CL20, Pik04] and the discussion in [Cur+21].

It is natural to ask why Alon's [Alo85] one-paragraph proof of Theorem 1.8.4 for $s = 2$ is hard to extend to $s > 2$.[13] Perhaps the simplest reason is that one cannot hope to show that in these cases the function $wsat(n, H)$ is subadditive (and then apply Fekete's lemma) since a subadditive function is necessarily of order $O(n)$, while we know from (1.6) that when $s \geq 3$ the function $wsat(n, H)$ is of order at least n^2. One can of course try to come up with more complicated recursive relations for $wsat(n, H)$ and combine them with variants of Fekete's lemma, but this seems to lead to a dead-end (we have certainly tried to go down that road). Our main novelty here is in finding a direct and efficient way to use an m-vertex r-graph witnessing the fact that $wsat(m, H)$ is small, in order to build arbitrarily large n-vertex r-graphs witnessing the fact that $wsat(n, H)$ is small. One of the main tools we use to construct such an example is Rödl's approximate designs theorem [Röd85] which enables us to efficiently combine many examples of size m into one of size n. Rödl's result would only allow us to construct a saturation process generating part of the edges of K_n^r. To complete this saturation process we would also need another set of gadgets.

13 While formally [Alo85] only deals with $r = 2$, the proof very similarly applies to $s = 2$ for arbitrary r.

[A G12] A. Grzesik. "On the maximum number of C_5's in a triangle-free graph." In: *Journal of Combinatorial Theory Ser. B* 102.5 (2012), pp. 1061–1066.

[AM21] A. Shapira and M. Tyomkyn. "A Ramsey variant of the Brown-Erdos-Sos conjecture." In: *Bulletin of the London Mathematical Society* 53.5 (2021), pp. 1453–1469.

[AM22] A. Shapira and M. Tyomkyn. *Weakly saturated hypergraphs and a conjecture of Tuza*. To appear in Proceedings of the AMS. 2022.

[A T87] A. Thomason. "Pseudorandom graphs." In: *Random graphs '85 (Poznan, 1985)*. Vol. 144. North-Holland Math. Stud. Amsterdam: North-Holland, 1987, pp. 307–331.

[A T89] A. Thomason. "A disproof of a conjecture of Erdos in Ramsey theory." In: *J. London Math. Soc* 39 (1989), pp. 246–255.

[AK78] R. Ahlswede and G. Katona. "Graphs with maximal number of adjacent pairs of edges." In: *Acta Mathematica Academiae Scientiarum Hungaricae* 32.1-2 (1978), pp. 97–120.

[Alo85] N. Alon. "An extremal problem for sets with applications to graph theory." In: *J. Combin. Theory Ser. A* 40.1 (1985), pp. 82–89.

[Alo+00] N. Alon, E. Fischer, M. Krivelevich, and M. Szegedy. "Efficient Testing of Large Graphs." English. In: *Combinatorica* 20.4 (2000), pp. 451–476.

[AGK04] N. Alon, G. Gutin, and M. Krivelevich. "Algorithms with large domination ratio." In: *Journal of Algorithms* 50.1 (2004), pp. 118–131.

[APS17] N. Alon, A. Pokrovskiy, and B. Sudakov. "Random subgraphs of properly edge-coloured complete graphs and long rainbow cycles." In: *Israel Journal of Mathematics* 222.1 (2017), pp. 317–331.

[AS04] N. Alon and J.H. Spencer. *The probabilistic method*. John Wiley and Sons, 2004.

[BCS86] B. Bollobas, C. Nara, and S. Tachibana. "The maximal number of induced complete bipartite graphs." In: *Discrete Mathematics* 62 (1986), pp. 271–275.

[B B+95] B. Bollobas, Y. Egawa, A. Harris, and G. P. Jin. "The maximal number of induced r-partite subgraphs." In: *Graphs and Combinatorics* 11.1 (1995), pp. 1–19.

[B N+09] B. Nagle, A. Poerschke, V. Rodl, and M. Schacht. "Hypergraph regularity and quasi-randomness." In: *Proceedings of the Twentieth Annual ACM-SIAM Symposium on Discrete Algorithms, SIAM, Philadelphia, PA*. 2009, pp. 227–235.

[BF88] L. Babai and P. Frankl. *Linear algebra methods in combinatorics*. University of Chicago, 1988.

[BPS17] I. Balla, A. Pokrovskiy, and B. Sudakov. "A remark on Hamilton cycles with few colors." In: *Moscow J. Combinatorics and Number Theory* 7 (2017), pp. 259–263.

[Bal+12] J. Balogh, B. Bollobás, R. Morris, and O.Riordan. "Linear algebra and bootstrap percolation." In: *J. Combin. Theory Ser. A* 119.6 (2012), pp. 1328–1335.

[BP98] J. Balogh and G. Pete. "Random disease on the square grid." In: *Proceedings of the Eighth International Conference "Random Structures and Algorithms" (Poznan, 1997)*. Vol. 13. 3-4. 1998, pp. 409–422.

[Beh18] N.C. Behague. "Hypergraph saturation irregularities." In: *The Electronic Journal of Combinatorics* 25 (2 2018).

[BPS20] F. Benzing, A. Pokrovskiy, and B. Sudakov. "Long directed rainbow cycles and rainbow spanning trees." In: *European Journal of Combinatorics* 88 (2020), p. 103102.

[Blo90] A. Blokhuis. "Solution of an extremal problem for sets using resultants of polynomials." In: *Combinatorica* 10.4 (1990), pp. 393–396.

[Bol65] B. Bollobás. "On generalized graphs." In: *Acta Mathematica Academiae Scientiarum Hungarica* 16 (1965), pp. 447–452.

[Bol68a] B. Bollobás. "Weakly k-saturated graphs." In: *Beitrage zur Graphentheorie (Kolloquium, Manebach, 1967)*. Teubner, Leipzig, 1968, pp. 25–31.

[Bol68b] B. Bollobás. "Weakly k-saturated graphs." In: *Beitrage zur Graphentheorie (Kolloquium, Manebach, 1967)*. Teubner, Leipzig, 1968, pp. 25–31.

[Bol86] B. Bollobás. *Combinatorics: Set Systems, Hypergraphs, Families of Vectors, and Combinatorial Probability*. Cambridge University Press, 1986.

[Bol98] B. Bollobás. *Modern graph theory*. Vol. 184. Springer Science and Business Media, 1998.

[BS02] M. Borowiecki and E. Sidorowicz. "Weakly P-saturated graphs." In: vol. 22. 1. Conference on Graph Theory (Elgersburg, 2000). 2002, pp. 17–29.

[BES73a] W. Brown, P. Erdős, and V. Sós. "On the existence of triangulated spheres in 3-graphs, and related problems." In: *Periodica Mathematica Hungarica* 3.3-4 (1973), pp. 221–228.

[BES73b] W. Brown, P. Erdős, and V. Sós. "Some extremal problems on *r*-graphs." In: New York: Academic Press, 1973, pp. 55–63.

[Buk15] B. Bukh. "Random algebraic construction of extremal graphs." In: *Bull. Lond. Math. Soc.* 47 (2015), pp. 939–945.

[BC18] B. Bukh and D. Conlon. "Rational exponents in extremal graph theory." In: *Journal of the European Mathematical Society* 20.7 (2018), pp. 1747–1757.

[BT20] B. Bukh and M. Tait. "Turan numbers of theta graphs." In: *Combin. Probab. Comput.* 29 (2020), pp. 495–507.

[BGT21] D. Bulavka, A. Goodarzi, and M. Tancer. "Optimal bounds for the colorful fractional Helly theorem." In: *37th International Symposium on Computational Geometry (SoCG 2021)*. Vol. 189. Leibniz International Proceedings in Informatics (LIPIcs). 2021.

[CVM18] C. Reiher, V. Rodl, and M. Schacht. "On a Turan problem in weakly quasirandom 3-uniform hypergraphs." In: *Journal of the European Mathematical Society* 20 (2018), pp. 1139–1159.

[C W20] C. Wong. "On the existence of dense substructures in finite groups." In: *Discrete Mathematics* 343.10 (2020).

[CL20] D. Chakraborti and P-S. Loh. "Minimizing the numbers of cliques and cycles of fixed size in an F-saturated graph." In: *European Journal of Combinatorics* 90 (2020).

[CGW89] F. Chung, R. Graham, and R. Wilson. "Quasi-random graphs." In: *Combinatorica* 9.4 (1989), pp. 345–362.

[Con19] D. Conlon. "Graphs with few paths of prescribed length between any two vertices." In: *Bulletin of the London Mathematical Society* 51.6 (2019), pp. 1015–1021.

[CL21] D. Conlon and J. Lee. "On the extremal number of subdivisions." In: *International Mathematics Research Notices* 2021.12 (2021), pp. 9122–9145.

[CLJ21] D. Conlon, J. Lee, and O. Janzer. "More on the extremal number of subdivisions." In: *Combinatorica* 41.4 (2021), pp. 465–494.

[CP20] M. Coulson and G. Perarnau. "A rainbow Dirac's theorem." In: *SIAM Journal on Discrete Mathematics* 34.3 (2020), pp. 1670–1692.

[Cur+21] B. Currie, J. Faudree, R. Faudree, and J. Schmitt. “A survey of minimum saturated graphs.” In: *The Electronic Journal of Combinatorics* 1000 (2021), DS19–Oct.

[DM21a] M. Tancer D. Bulavka and M. Tyomkyn. *Weak saturation of multipartite hypergraphs*. Accepted to Combinatorica, subject to minor revision. Preprint available at arxiv:2109.03703. 2021.

[DJ13] D. Conlon and J. Fox. “Graph removal lemmas.” In: *Surveys in Combinatorics*. Cambridge University Press, 2013, pp. 1–50.

[DJB10] D. Conlon, J. Fox, and B. Sudakov. “Hypergraph Ramsey numbers.” In: *J. Amer. Math. Soc* 23 (2010), pp. 247–266.

[DJB11] D. Conlon, J. Fox, and B. Sudakov. “Large almost monochromatic subsets in hypergraphs.” In: *Isr. J. Math.* 181 (2011), pp. 423–432.

[DJB12] D. Conlon, J. Fox, and B. Sudakov. “Erdos–Hajnal-type theorems in hypergraphs.” In: *Journal Comb. Theory, Series B* 102 (2012), pp. 1142–1154.

[D C+] D. Conlon, L. Gishboliner, Y. Levanzov, and A. Shapira. “A new bound for the Brown–Erdos–Sos problem.” In: *J. Combin. Theory Ser. B* ().

[DM21b] D. Conlon and M. Tyomkyn. “Repeated patterns in proper edge-colourings.” In: *SIAM J. Discrete Math* 35.3 (2021), pp. 2249–2264.

[DM23] D. Conlon and M. Tyomkyn. “Ramsey numbers of trails and circuits.” In: *Journal of Graph Theory* 102.1 (2023), pp. 194–196.

[DM22] S. Luo D. Conlon and M. Tyomkyn. *Monochromatic components with many edges*. Accepted to Journal of Combinatorics. Preprint available at arxiv:2204.11360. 2022.

[DM16] D. Hefetz and M. Tyomkyn. “Universality of graphs with few triangles and anti-triangles.” In: *Combinatorics, Probability and Computing* 25.4 (2016), pp. 560–576.

[DM17] D. Hefetz and M. Tyomkyn. “On the inducibility of cycles.” In: *arXiv preprint arxiv:1702.07342* (2017).

[DM18] D. Hefetz and M. Tyomkyn. “On the inducibility of cycles.” In: *J. Combinatorial Theory, Ser. B* 133 (2018), pp. 243–258.

[DP13] D. Hefetz and P. Keevash. “A hypergraph Turan theorem via lagrangians of intersecting families.” In: *J. Combin. Theory Ser. A* 120 (2013), pp. 2020–2038.

[DJ17] D. Solymosi and J. Solymosi. “Small cores in 3-uniform hypergraphs.” In: *Journal of Combinatorial Theory, Series B* 122 (2017), pp. 897–910.

[DeB+21] L. DeBiasio, Y. Kamel, G. McCourt, and H. Sheats. “Generalizations and strengthenings of Ryser’s conjecture.” In: *Electron. J. Combin.* 28.4 (2021), Paper No. 4.37.

[DKS20] L. DeBiasio, R. Krueger, and G. Sárközy. “Large monochromatic components in multicolored bipartite graphs.” In: *J. Graph Theory* 94.1 (2020), pp. 117–130.

[E S78] E. Szemerédi. “Regular partitions of graphs.” In: *Proc. Colloque Inter. CNRS*. 1978, pp. 399–401.

[EGJ20] S. Ehard, S. Glock, and F. Joos. “A rainbow blow-up lemma for almost optimally bounded edge-colourings.” In: 8 (2020), e37.

[Erd45] P. Erdős. “On a lemma of Littlewood and Offord.” In: *Bulletin of the American Mathematical Society* 51.12 (1945), pp. 898–902.

[Erd81a] P. Erdős. “On the combinatorial problems which I would most like to see solved.” In: *Combinatorica* 1.1 (1981), pp. 25–42.

[Erd81b] P. Erdős. “Problems and results in graph theory.” In: *The theory and applications of graphs (Kalamazoo, MI, 1980)* (1981), pp. 331–341.

[Erd81c] P. Erdős. “Some new problems and results in graph theory and other branches of combinatorial mathematics.” In: *Combinatorics and Graph Theory*. 1981, pp. 9–17.

[Erd83] P. Erdős. “On Some of My Conjectures in Number Theory and Combinatorics.” In: *Proceedings of the fourteenth Southeastern conference on combinatorics, graph theory and computing*. Vol. 39. 3-4. Boca Raton, 1983, pp. 3–19.

[Erd84] P. Erdős. “On some problems in graph theory, combinatorial analysis and combinatorial number theory.” In: *Graph theory and combinatorics (Cambridge, 1983)* (1984), pp. 1–17.

[Erd+97] P. Erdős, R. Faudree, R. Gould, M. Jacobson, and J. Lehel. “Edge disjoint monochromatic triangles in 2-colored graphs.” In: *Discrete Math* 164 (1997), pp. 81–85.

[EFT91] P. Erdős, Z. Füredi, and Z. Tuza. “Saturated r-uniform hypergraphs.” In: *Discrete Math.* 98.2 (1991), pp. 95–104.

[EHM64] P. Erdős, A. Hajnal, and J. W. Moon. “A Problem in Graph Theory.” In: *The American Mathematical Monthly* 71.10 (1964), pp. 1107–1110.

[EL15] C. Even-Zohar and N. Linial. "A Note on the Inducibility of 4-Vertex Graphs." In: *Graphs and Combinatorics* 31.5 (2015), pp. 1367–1380.

[FR89] F. Chung and R. Graham. "Quasi-random hypergraphs." In: *Proc. Natl.. Acad. Sci.* 86 (1989), pp. 8175–8177.

[FV13] V. Falgas-Ravry and E. Vaughan. "Applications of the semi-definite method to the Turan density problem for 3-graphs." In: *Combin. Probab. Comput.* 22.1 (2013), pp. 21–54.

[FG14] R. Faudree and R. Gould. "Weak saturation numbers for multiple copies." In: *Discrete Math.* 336 (2014), pp. 1–6.

[FGJ13] R. Faudree, R. Gould, and M. Jacobson. "Weak saturation numbers for sparse graphs." In: *Discuss. Math. Graph Theory* 33.4 (2013), pp. 677–693.

[FS83] R. Faudree and M. Simonovits. "On a class of degenerate extremal graph problems." In: *Combinatorica* 3.1 (1983), pp. 83–93.

[Fei06] U. Feige. "On sums of independent random variables with unbounded variance and estimating the average degree in a graph." In: *SIAM Journal on Computing* 35.4 (2006), pp. 964–984.

[Fek23] M. Fekete. "Uber die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten." In: *Mathematische Zeitschrift* 17.1 (1923), pp. 228–249.

[Fra82] P. Frankl. "An extremal problem for two families of sets." In: *European J. Combin.* 3.2 (1982), pp. 125–127.

[Für81] Z. Füredi. "Maximum degree and fractional matchings in uniform hypergraphs." In: *Combinatorica* 1.2 (1981), pp. 155–162.

[Für89] Z. Füredi. "Covering the complete graph by partitions." In: *Discrete Math.* 75.1-3 (1989), pp. 217–226.

[G T18] G. Tardos. "Extremal theory of ordered graphs." In: *Proceedings of the International Congress of Mathematics*. Vol. 3. 2018, pp. 3219–3228.

[Gan98] F. Gantmacher. *The theory of matrices. Vol. 1*. Translated from the Russian by K. A. Hirsch, Reprint of the 1959 translation. AMS Chelsea Publishing, Providence, RI, 1998, pp. x+374. ISBN: 0-8218-1376-5.

[Gao+21] P. Gao, R. Ramadurai, I. Wanless, and N. Wormald. "Full rainbow matchings in graphs and hypergraphs." In: *Combinatorics, Probability and Computing* 30.5 (2021), pp. 762–780.

[GJ20] S. Glock and F. Joos. "A rainbow blow-up lemma." In: *Random Structures and Algorithms* 56.4 (2020), pp. 1031–1069.

[Glo+21] S. Glock, D. Kühn, R. Montgomery, and D. Osthus. "Decompositions into isomorphic rainbow spanning trees." In: *Journal of Combinatorial Theory, Series B* 146 (2021), pp. 439–484.

[Goo59] A. W. Goodman. "On sets of acquaintances and strangers at any party." In: *Amer. Math. Monthly* 66 (1959), pp. 778–783.

[GRS91] R. Graham, B. Rothschild, and J. Spencer. *Ramsey Theory*. Wiley Series in Discrete Mathematics and Optimization. Wiley, 1991.

[GJN22] A. Grzesik, O. Janzer, and Z.L. Nagy. "The Turan number of blow-ups of trees." In: *Journal of Combinatorial Theory, Series B* 156 (2022), pp. 299–309.

[Gus91] T. Gustavsson. "Decompositions of large graphs and digraphs with high minimum degree." In: *Doctoral Dissertation, Dept. of Mathematics, Univ. of Stockholm*. 1991.

[Gyá77] A. Gyárfás. "Partition coverings and blocking sets in hypergraphs." In: *Communications of the Computer and Automation Institute of the Hungarian Academy of Sciences* 71 (1977), 62 pp.

[Gyá11] A. Gyárfás. "Large monochromatic components in edge colorings of graphs: a survey." In: *Ramsey theory*. Vol. 285. Progr. Math. Birkhauser/Springer, New York, 2011, pp. 77–96.

[GS17] A. Gyárfás and G. Sárközy. "Large monochromatic components in edge colored graphs with a minimum degree condition." In: *Electron. J. Combin.* 24.3 (2017), Paper No. 3.54.

[Győ89] E. Győri. "On the number of C5's in a triangle-free graph." In: *Combinatorica (Print)* 9.1 (1989), pp. 101–102.

[Hat+13] H. Hatami, J. Hladky, D.Kral, S. Norine, and A. Razborov. "On the number of pentagons in triangle-free graphs." In: *Journal of Combinatorial Theory, Series A* 120.3 (2013), pp. 722–732.

[Hir14] J. Hirst. "The inducibility of graphs on four vertices." In: *Journal of Graph Theory* 75.3 (2014), pp. 231–243.

[Hua14] H. Huang. "On the maximum induced density of directed stars and related problems." In: *SIAM Journal on Discrete Mathematics* 28.1 (2014), pp. 92–98.

[J B+16] J. Balogh, P. Hu, B. Lidicky, and F. Pfender. "Maximum density of induced 5-cycle is achieved by an iterated blow-up of 5-cycle." In: *European Journal of Combinatorics* 52 (2016), pp. 47–58.

[JA94] J. Brown and A. Sidorenko. "The inducibility of complete bipartite graphs." In: *Journal of Graph Theory* 18.6 (1994), pp. 629–645.

[JB08] J. Fox and B. Sudakov. "Induced Ramsey-type theorems." In: *Adv. Math.* 219 (2008), pp. 1771–1800.

[JL20] J. Fox and L. Sauermann. "A completion of the proof of the Edge-statistics Conjecture." In: *Advances in Combinatorics* 4 (2020).

[J L20] J. Long. "A note on the Brown-Erdos-Sos conjecture in groups." In: *Combinatorics, Probability and Computing* 29.4 (2020), pp. 633–640.

[JC20] J. Solymosi and C. Wong. "The Brown-Erdos-Sos Conjecture in finite abelian groups." In: *Discret. Appl. Math.* 276 (2020), pp. 155–160.

[J T02] J. Talbot. "Lagrangians of hypergraphs." In: *Combin., Probab. Comput.* 11 (2002), pp. 199–216.

[JLR11] Svante Janson, Tomasz Luczak, and Andrzej Rucinski. *Random graphs*. John Wiley and Sons, 2011.

[KSC11] J. Kahn, M. Saks, and C.Smyth. "The dual BKR inequality and rudich's conjecture." In: *Combinatorics, Probability and Computing* 20.2 (2011), pp. 257–266.

[Kal84a] G. Kalai. "Intersection patterns of convex sets." In: *Israel J. Math.* 48.2-3 (1984), pp. 161–174.

[Kal84b] G. Kalai. "Weakly saturated graphs are rigid." In: *Convexity and graph theory (Jerusalem, 1981)*. Vol. 87. North-Holland Math. Stud. North-Holland, Amsterdam, 1984, pp. 189–190.

[Kal85] G. Kalai. "Hyperconnectivity of graphs." In: *Graphs Combin.* 1.1 (1985), pp. 65–79.

[KNS64] G. Katona, T. Nemetz, and M. Simonovits. "On a problem of Turan in the theory of graphs." In: *Mat. Lapok* 15 (1964), pp. 228–238.

[Kee11] P. Keevash. "Hypergraph Turan problems." In: *Surveys in Combinatorics 2011*. Ed. by RobinEditor Chapman. London Mathematical Society Lecture Note Series. Cambridge University Press, 2011, pp. 83–140.

[KY18] P. Keevash and L. Yepremyan. "Rainbow Matchings in Properly Colored Multigraphs." In: *SIAM Journal on Discrete Mathematics* 32.3 (2018), pp. 1577–1584.

[Kim+20] J. Kim, D. Kühn, A. Kupavskii, and D. Osthus. "Rainbow structures in locally bounded colorings of graphs." In: *Random Structures and Algorithms* 56.4 (2020), pp. 1171–1204.

[KNV19] D. Kral, S. Norin, and J. Volec. "A bound on the inducibility of cycles." In: *Journal of Combinatorial Theory, Series A* 161 (2019), pp. 359–363.

[KMM21] G. Kronenberg, T. Martins, and N. Morrison. "Weak saturation numbers of complete bipartite graphs in the clique." In: *J. Combin. Theory Ser. A* 178 (2021), pp. 105357, 15.

[KST19] M. Kwan, B. Sudakov, and T. Tran. "Anticoncentration for subgraph statistics." In: *Journal of the London Mathematical Society* 99.3 (2019), pp. 757–777.

[LM15] N. Linial and A. Morgenstern. "Graphs with Few 3-Cliques and 3-Anticliques are 3-Universal." In: *Journal of Graph Theory* 78.3 (2015), pp. 229–238.

[Lov77] L. Lovász. "Flats in matroids and geometric graphs." In: *Combinatorial surveys (Proc. Sixth British Combinatorial Conf., Royal Holloway Coll., Egham, 1977)*. 1977, pp. 45–86.

[Luo] S. Luo. "On connected components with many edges." In: *Preprint available at arXiv:2111.13342 [math.CO]* ().

[MM21] A. Shapira M. Amir and M. Tyomkyn. "Two Erdos–Hajnal-type theorems in hypergraphs." In: *J. Combinatorial Theory, Ser. B* 146 (2021), pp. 417–438.

[M C14] M. Chudnovsky. "The Erdos–Hajnal conjecture – A survey." In: *J. of Graph Theory* 75 (2014), pp. 178–190.

[M T17] M. Tyomkyn. "Lagrangians of hypergraphs: The Frankl-Furedi conjecture holds almost everywhere." In: *Journal of the London Mathematical Society* 96.3 (2017), pp. 584–600.

[M T21] M. Tyomkyn. "Many disjoint triangles in co-triangle-free graphs." In: *Combinatorics, Probability and Computing* 30.1 (2021), pp. 153–162.

[Mar+19] A. Martinsson, F. Mousset, A. Noever, and M. Trujic. "The edge-statistics conjecture for $\ell \ll k^{6/5}$." In: *Israel Journal of Mathematics* 234.2 (2019), pp. 677–690.

[MPS19] R. Montgomery, A. Pokrovskiy, and B. Sudakov. "Decompositions into spanning rainbow structures." In: *Proceedings of the London Mathematical Society* 119.4 (2019), pp. 899–959.

[MPS20] R. Montgomery, A. Pokrovskiy, and B. Sudakov. "Embedding rainbow trees with applications to graph labelling and decomposition." In: *Journal of the European Mathematical Society* 22 (2020), pp. 3101–3132.

[MPS21] R. Montgomery, A. Pokrovskiy, and B. Sudakov. "A proof of Ringel's conjecture." In: *Geom. Funct. Anal.* 31 (2021), pp. 663–720.

[MN18] N. Morrison and J. Noel. "Extremal bounds for bootstrap percolation in the hypercube." In: *J. Combin. Theory Ser. A* 156 (2018), pp. 61–84.

[MS17] N. Morrison and A. Scott. "Maximising the number of induced cycles in a graph." In: *Journal of Combinatorial Theory, Series B* 126 (2017), pp. 24–61.

[MS15] G. Moshkovitz and A. Shapira. "Exact bounds for some hypergraph saturation problems." In: *J. Combin. Theory Ser. B* 111 (2015), pp. 242–248.

[NAB09] N. Alon, A. Shapira, and B. Sudakov. "Additive approximation for edge-deletion problems." In: *Annals of Math* 170 (2009), pp. 371–411.

[NJJ01] N. Alon, J. Pach, and J. Solymosi. "Ramsey-type theorems with forbidden subgraphs." In: *Combinatorica* 21 (2001), pp. 155–170.

[NM20] M. Krivelevich N. Alon D. Hefetz and M. Tyomkyn. "Edge-statistics on large graphs." In: *Combinatorics, Probability and Computing* 29.2 (2020), pp. 163–189.

[O J19] O. Janzer. "Improved bounds for the extremal number of subdivisions." In: *Electron. J. Combin.* 26.P3.3 (2019).

[O J] O. Janzer. "Rainbow Turan number of even cycles, repeated patterns and blow-ups of cycles." In: *Israel J. Math.* ().

[OV18] J. O'Neill and J. Verstraete. "A generalization of the Bollobas set pairs inequality." In: *arXiv preprint arXiv:1812.00537* (2018).

[Osu21] M. Osumi. "Ramsey numbers of trails." In: *arXiv preprint arXiv:2109.00734* (2021).

[P E62] P. Erdős. "On the number of complete subgraphs contained in certain graphs." In: *Magyar Tud. Akad. Mat. Kulato Int. Kozl* 7 (1962), pp. 459–464.

[P E64] P. Erdős. "On extremal problems of graphs and generalized graphs." In: *Isr. J. Math.* 2 (1964), pp. 183–190.

[P E75] P. Erdős. "Problems and results in combinatorial number theory." In: *Journees arithmetiques de Bordeaux*. Vol. 24–25. Asterisque, 1975, pp. 295–310.

[P E76] P. Erdős. "Problems and results in combinatorial analysis." In: *Colloquio Internazionale sulle Teorie Combinatorie, 1973*. Rome: Accademia Nazionale dei Lincei, 1976, pp. 3–17.

[P E97] P. Erdős. "Some recent problems and results in graph theory." In: *Discrete Math* 164 (1997), pp. 81–85.

[PA89] P. Erdős and A. Hajnal. "Ramsey-type theorems." In: *Discrete Applied Mathematics* 25.1–2 (1989), pp. 37–52.

[PR52] P. Erdős and R. Rado. "Combinatorial theorems on classifications of subsets of a given set." In: *Proc. London Math. Soc.* 3 (1952), pp. 417–439.

[PV84] P. Frankl and V. Rodl. "Hypergraphs do not jump." In: *Combinatorica* 4 (1984), pp. 149–159.

[PZ89] P. Frankl and Z. Füredi. "Extremal problems whose solutions are the blow-ups of the small Witt-designs." In: *J. Combin. Theory Ser. A* 52 (1989), pp. 129–147.

[PV01] P. Haxell and V. Rodl. "Integer and fractional packings in dense graphs." In: *Combinatorica* 21 (2001), pp. 13–38.

[PB04] P. Keevash and B. Sudakov. "Packing triangles in a graph and its complement." In: *Journal of Graph theory* 47.3 (2004), pp. 203–216.

[PJ] P. Keevash and J. Long. "The Brown-Erdos-Sos Conjecture for hypergraphs of large uniformity." In: *Proc. Amer. Math. Soc.* ().

[PK20] P. Keevash and K. Staden. "Ringel's tree packing conjecture in quasirandom graphs." In: *arXiv preprint arXiv:204.09947* (2020).

[Pik99] O. Pikhurko. "The Minimum Size of Saturated Hypergraphs." In: *Combinatorics, Probability and Computing* 8.5 (1999), pp. 483–492.

[Pik01a] O. Pikhurko. "Uniform families and count matroids." In: *Graphs Combin.* 17.4 (2001), pp. 729–740.

[Pik01b] O. Pikhurko. "Weakly saturated hypergraphs and exterior algebra." In: *Combin. Probab. Comput.* 10.5 (2001), pp. 435–451.

[Pik04] O. Pikhurko. "Results and Open Problems on Minimum Saturated Hypergraphs." In: *Ars Comb.* 72 (July 2004), pp. 111–127.

[Pin83] J.-E. Pin. "On two combinatorial problems arising from automata theory." In: *North-Holland Mathematics Studies*. Vol. 75. Elsevier, 1983, pp. 535–548.

[PG75] N. Pippenger and M. Golumbic. "The inducibility of graphs." In: *Journal of Combinatorial Theory, Series B* 19.3 (1975), pp. 189–203.

[Pok18] A. Pokrovskiy. "An approximate version of a conjecture of Aharoni and Berger." In: *Adv. Math.* 333 (2018), pp. 1197–1241.

[PS18] A. Pokrovskiy and B. Sudakov. "Linearly many rainbow trees in properly edge-coloured complete graphs." In: *J. Combin. Theory Ser. B* 132 (2018), pp. 134–156.

[PS19] A. Pokrovskiy and B. Sudakov. "A counterexample to Stein's equi-n-square conjecture." In: *Proc. Amer. Math. Soc.* 147 (2019), pp. 2281–2287.

[Q T+16] Q. Tang, Y. Peng, X. Zhang, and C. Zhao. "Connection between the clique number and the Lagrangian of 3-uniform hypergraphs." In: *Optimization Letters* 10.4 (2016), pp. 685–697.

[R K20] R. Krueger. "Generalized Ramsey numbers: forbidding paths with few colors." In: *Electron. J. Combin.* 27.P1.44 (2020).

[R M75] R. M. Wilson. "An existence theory for pairwise balanced designs. III. Proof of the existence conjectures." In: *J. Combin. Theory Ser. A* 18 (1975), pp. 71–79.

[RBM20] R. Nenadov, B. Sudakov, and M. Tyomkyn. "Proof of the Brown-Erdos-Sos conjecture in groups." In: *Mathematical Proceedings of the Cambridge Philosophical Society* 169.2 (2020), pp. 323–333.

[R Y07] R. Yuster. "Packing cliques in graphs with independence number 2." In: *Combin. Probab. Comput.* 16.5 (2007), pp. 805–817.

[Raz07] A. Razborov. "Flag algebras." In: *The Journal of Symbolic Logic* 72.4 (2007), pp. 1239–1282.

[Röd85] V. Rödl. "On a Packing and Covering Problem." In: *European Journal of Combinatorics* 6.1 (1985), pp. 69–78.

[Ros19] A. Rosen. *Geometric multivector analysis*. Birkhauser Advanced Texts: Basler Lehrbucher. From Grassmann to Dirac. Birkhauser/Springer, Cham, 2019, pp. xii+465.

[Rot53] K. Roth. "On certain sets of integers." In: *Journal of the London Mathematical Society* 1.1 (1953), pp. 104–109.

[Rus12] M. Ruszinko. "Large components in r-edge-colorings of K_n have diameter at most five." In: *J. Graph Theory* 69.3 (2012), pp. 337–340.

[RS78] I. Ruzsa and E. Szemerédi. "Triple systems with no six points carrying three triangles." In: *Combinatorics (Keszthely, 1976), Coll. Math. Soc. J. Bolyai* 18 (1978), pp. 939–945.

[S G+20] S. Glock, D. Kuhn, A. Lo, and D. Osthus. "On a conjecture of Erdos on locally sparse Steiner triple systems." In: *Combinatorica* 40 (2020), pp. 363–403.

[SA54] S. Lang and A. Weil. "NUMBER OF POINTS OF VARIETIES IN FINITE FIELDS." In: *American Journal of Mathematics* 76 (1954), pp. 819–827.

[S R11] S. Radziszowski. "Small Ramsey numbers." In: *Electronic Jounal of Combinatorics* DS1 (2011).

[SS04] G. Sárközy and S. Selkow. "An extension of the Ruzsa-Szemeredi theorem." In: *Combinatorica* 25.1 (2004), pp. 77–84.

[SW19] A. Scott and E. Wilmer. "Combinatorics in the exterior algebra and the Bollobas Two Families Theorem." In: *arXiv preprint arXiv:1907.06019* (2019).

[Sei74] D. Seinsche. "On a property of the class of n-colorable graphs." In: *J. Combinatorial Theory Ser. B* 16 (1974), pp. 191–193.

[Sem97] G. Semanisin. "On some variations of extremal graph problems." In: *Discuss. Math. Graph Theory* 17.1 (1997), pp. 67–76.

[Sid07] E. Sidorowicz. "Size of weakly saturated graphs." In: *Discrete Math.* 307.11-12 (2007), pp. 1486–1492.

[Sim68] M. Simonovits. "A method for solving extremal problems in graph theory, stability problems." In: *Theory of Graphs (Proc. Colloq., Tihany, 1966)*. New York: Academic Press, 1968, pp. 279–319.

[Sol15] J. Solymosi. "The (7, 4)-conjecture in finite groups." In: *Combinatorics, Probability and Computing* 24.4 (2015), pp. 680–686.

[Sze13] V. Szecsi. "On the minimal length of the longest trail in a fixed edge-density graph." In: *Cent. Eur. J. Math.* 11.10 (2013), pp. 1831–1837.

[Sze75] E. Szemerédi. "On sets of integers containing no k elements in arithmetic progression." In: *Acta Arith.* 27 (1975), pp. 199–245.

[TL19] T. Bohman and L. Warnke. "Large girth approximate Steiner triple systems." In: *J. Lond. Math. Soc.* 100.3 (2019), pp. 895–913.

[TVP54] T. Kovari, V. Sós, and P. Turán. "On a problem of K. Zarankiewicz." In: *Colloq. Math.* 3 (1954), pp. 50–57.

[TE65] T. Motzkin and E. Strauss. "Maxima for graphs and a new proof of a theorem of Turan." In: *Canad. J. Math* 17 (1965), pp. 533–540.

[Tuz86] Z. Tuza. "A generalization of saturated graphs for finite languages." In: *Tanulmanyok—MTA Szamitastech. Automat. Kutato Int. Budapest,(185)* (1986), pp. 287–293.

[Tuz88] Z. Tuza. “Extremal problems on saturated graphs and hypergraphs.” In: vol. 25. B. Eleventh British Combinatorial Conference (London, 1987). 1988, pp. 105–113.

[Tuz92] Z. Tuza. “Asymptotic growth of sparse saturated structures is locally determined.” In: *Discrete Math.* 108.1-3 (1992), pp. 397–402.

[VS20a] V. Gruslys and S. Letzter. “Fractional triangle decompositions in almost complete graphs.” In: *arXiv preprint arXiv:2008.05313* (2020).

[VS20b] V. Gruslys and S. Letzter. “Monochromatic triangle packings in red-blue graphs.” In: *arXiv preprint arXiv:2008.05311* (2020).

[VSN20] V. Gruslys, S. Letzter, and N. Morrison. “Hypergraph Lagrangians I: the Frankl-Füredi conjecture is false.” In: *Adv. Math.* 365 (2020), p. 107063.

[VSN] V. Gruslys, S. Letzter, and N. Morrison. “Hypergraph Lagrangians II: when colex is best.” In: *Israel J. Math.* ().

[V N08] V. Nikiforov. “Graphs with many r-cliques have large complete r-partite subgraphs.” In: *Bull. Lond. Math. Soc.* 40 (2008), pp. 23–25.

[V R86] V. Rodl. “On universality of graphs with uniformly distributed edges.” In: *Discrete Math.* 59 (1986), pp. 125–134.

[VP89] V. Rodl and P. Winkler. “A Ramsey-type theorem for orderings of a graph.” In: *SIAM Journal of Discrete Mathematics* 2 (1989), pp. 402–406.

[VS12] V. Rodl and M. Schacht. “Complete partite subgraphs in dense hypergraphs.” In: *Random Structures and Algorithms* 41 (2012), pp. 557–573.

[Wae27] B. L. van der Waerden. “Beweis einer Baudetschen Vermutung.” In: *Nieuw. Arch. Wisk.* 15 (1927), pp. 212–216.

[Yus19] R. Yuster. “On the exact maximum induced density of almost all graphs and their inducibility.” In: *Journal of Combinatorial Theory, Series B* 136 (2019), pp. 81–109.

[ZR13] Z. Füredi and M. Ruszinko. “Uniform hypergraphs containing no grids.” In: *Adv. Math.* 240 (2013), pp. 302–324.

[Zyk49] A. Zykov. “On some properties of linear complexes.” In: *Matematicheskii sbornik* 66.2 (1949). In Russian, pp. 163–188.

2

LAGRANGIANS OF HYPERGRAPHS

This chapter contains the following publication:

REFERENCES

[M T17] M. Tyomkyn. “Lagrangians of hypergraphs: The Frankl-Furedi conjecture holds almost everywhere.” In: *Journal of the London Mathematical Society* 96.3 (2017), pp. 584–600.

3 THREE RAMSEY RESULTS FOR UNIFORM HYPERGRAPHS

This chapter contains the following publications:

REFERENCES

[AM21] A. Shapira and M. Tyomkyn. "A Ramsey variant of the Brown-Erdos-Sos conjecture." In: *Bulletin of the London Mathematical Society* 53.5 (2021), pp. 1453–1469.

[MM21] A. Shapira M. Amir and M. Tyomkyn. "Two Erdos–Hajnal-type theorems in hypergraphs." In: *J. Combinatorial Theory, Ser. B* 146 (2021), pp. 417–438.

4 INDUCIBILITY AND EDGE-STATISTICS

This chapter contains the following publications:

REFERENCES

[DM18] D. Hefetz and M. Tyomkyn. "On the inducibility of cycles." In: *J. Combinatorial Theory, Ser. B* 133 (2018), pp. 243–258.

[NM20] M. Krivelevich N. Alon D. Hefetz and M. Tyomkyn. "Edge-statistics on large graphs." In: *Combinatorics, Probability and Computing* 29.2 (2020), pp. 163–189.

5 TRIANGLES AND ANTI-TRIANGLES

This chapter contains the following publications:

REFERENCES

[DM16] D. Hefetz and M. Tyomkyn. "Universality of graphs with few triangles and anti-triangles." In: *Combinatorics, Probability and Computing* 25.4 (2016), pp. 560–576.

[M T21] M. Tyomkyn. "Many disjoint triangles in co-triangle-free graphs." In: *Combinatorics, Probability and Computing* 30.1 (2021), pp. 153–162.

6 MONOCHROMATIC CONNECTED COMPONENTS

This chapter contains the following publications:

REFERENCES

[DM23] D. Conlon and M. Tyomkyn. "Ramsey numbers of trails and circuits." In: *Journal of Graph Theory* 102.1 (2023), pp. 194–196.

[DM22] S. Luo D. Conlon and M. Tyomkyn. *Monochromatic components with many edges*. Accepted to Journal of Combinatorics. Preprint available at arxiv:2204.11360. 2022.

7 REPEATED PATTERNS IN PROPER COLOURINGS

This chapter contains the following publication:

REFERENCES

[DM21] D. Conlon and M. Tyomkyn. "Repeated patterns in proper edge-colourings." In: *SIAM J. Discrete Math* 35.3 (2021), pp. 2249–2264.

8

WEAK SATURATION IN GRAPHS AND HYPERGRAPHS

This chapter contains the following publications:

REFERENCES

[AM22] A. Shapira and M. Tyomkyn. *Weakly saturated hypergraphs and a conjecture of Tuza*. To appear in Proceedings of the AMS. 2022.

[DM21] M. Tancer D. Bulavka and M. Tyomkyn. *Weak saturation of multipartite hypergraphs*. Accepted to Combinatorica, subject to minor revision. Preprint available at arxiv:2109.03703. 2021.