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On the dimension reduction for partial differential equations

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1 Introduction

Partial differential equations (PDEs) belong among the natural and important parts of mathematics. Many of them have a strong physical background and their final forms often rise from continuum mechanics. It leads to widespread applications of PDEs. We must be, however, aware of the fact that PDEs are only approximations of reality. The approximations can be more or less suitable but in any case, many questions related to their solutions should be answered. The most important questions are related to the existence and uniqueness of solutions and their properties. Answers to the questions enable us to understand how good the equations are for solving real-world problems. The answers can be further used for predictions or optimizations. Other properties of the solutions are related to their regularity or various kinds of asymptotic behavior. The last category can provide us with information about hidden properties of the solutions, unknown connections, and the limits of usability of the respective equations.

The asymptotic behavior of the solutions often involves their behavior for time or space variables tending to infinity. Other problems are related to homogenization. In the habilitation thesis, we will, however, pay attention to a different type of asymptotic behavior, namely dimension reduction. The dimension reduction was done mostly intuitively and the reasoning was very simple. Let us assume we have a partial differential equation modeling a three-dimensional problem. If we have some additional information that in one or two spatial dimensions nothing happens we can simply cancel coordinates of corresponding vectors like velocity field and we get a partial differential equation with a lower number of spatial dimensions which is often easier to solve (in general, it is not so straightforward as we will see in the section about elasticity). However, the natural question arises: Can we get the same result if we follow a mathematically rigorous path? By the mathematically rigorous path, we mean a limit process such that some of the quantities or components of vectors converge to zero. If the answer is positive then we are in coincidence with our physical intuition. If not, then we have probably discovered a limiting factor for given partial differential equations or we have found out that our intuition mislead us. The next important thing is how boundary conditions in a higher-dimensional problem change respective to lower-dimensional equations. This kind of problem starts to be more complicated in the case of deformed domains. In this case, the deformations of the domains affect the limit process and limit equations. There are three approaches to tackling the problem. The most natural approach seems to be to simply estimate the difference between the three-dimensional solution and the solution of a given or known in advance lower-dimensional model. The second approach is based on the constrained method and the last one requires transforming the problem on a referential domain and then using techniques such as the formal series expansion, scaling, and a priori estimates.

The study of the dimension reduction process has also one important consequence in numerical mathematics. If you want to compute an approximate solution using a numerical approach then you can suffer a failure if one or two dimensions are much smaller than the next ones. However, in case you can control the dimension reduction process, you can use lower-dimensional models for the construction of approximate solutions without the disproportionate relations among dimensions. The usage of the lower-dimensional models can save a lot of computational time and the lower-dimensional models are often better theoretically understood.

There is also another problem, especially in elasticity. If you have a "lower-dimensional" elastic body, you have several available theories based on different geometrical or mechanical assumptions. But which of them is the most suitable for your elastic body? And what is its relation to the respective three-dimensional elastic model?

In the habilitation thesis, we want to present, how the dimension reduction approach can be used for the derivation of limit, lower-dimensional partial differential equations for elastic materials and fluids. We pay attention to more complicated cases where respective domains are somehow deformed. The two pieces of stuff selected for the thesis rank among the most important and the most studied parts of PDEs. It is necessary to say that there is no one equation for elastic materials and one equation for fluids. Both of them can be described by vast quantities of partial differential equations that are related to the properties of the material and to the kind of problem we want to solve. You can thus find linear and nonlinear equations, steady and non-steady equations, etc.. It means that there is no unique approach to the corresponding PDEs and many of them require special treatment and technique. Fortunately, the basic ideas related to dimension reduction seem to apply to many of the equations and the rest can be tuned in such a way that involves special properties of the studied equations.

In the habilitation thesis, we thus study two systems of equations related to elastic problems and one system of equations related to compressible fluids. A unifying factor of the thesis is the deformed domain where we study the systems. In the case of elastic materials, we pay attention to curved rods which leads to one-dimensional systems. In the case of fluids, we derive respective two-dimensional systems using shells as deformed domains. The paper consists of several parts. In Section 1, we give an overview of elasticity and fluids together with the current state of the art related to dimension reduction. In Section 2, we summarize the contribution of the author of the thesis to the topics related to dimension reduction. Section 3 deals with equations we want to study together with their boundary and initial conditions. We also suggest the main difficulties that must be overcome during our limit process. In Section 4, we introduce frameworks for deformed domains. We show how to describe the curved rods and shells using suitable mappings and referential straight domains requiring some kind of symmetry. We also introduce basic notation and function spaces together with a special function space related to our problems. We further mention basic inequalities and properties we will use in the next sections. In Section 5, we show how dimension reduction works for elastic problems. First, we pay attention to a dynamic linear model. Second, we study a dynamic nonlinear model involving heat. Section 6 contains an application of the dimension reduction to the Navier-Stokes equations for compressible, nonlinearly viscous fluids.

1.1 Elasticity

To study a real-world problem, it is often necessary to describe it by equations. One of the main sources of the equations is continuum mechanics. As introductory books related to elasticity, we can recommend the books [205], [216], and [89]. A more subtle approach can be found in [203] and [204]. We can of course find some more recent literature as [69] and [60]. A comprehensive introduction to elasticity and respective mathematical treatment can be found in [42]. The book can also serve as a good review of what was done in elasticity up to the year 1988. To go deeper in the mathematical treatment of elasticity we can also recommend the classical literature as [73] and [159]. Mathematically oriented treatments of nonlinear elasticity can be found in [163] and [16]. As we can see from the above-mentioned books, the amount of literature related to elasticity is huge despite the fact we have not mentioned anything about numerics, optimizations, etc. As the last introductory literature, we thus recommend the reader the book [181], where he can also find various applications of elasticity and an introduction to the numerical treatment. There are also various extensions of elasticity. One of them called thermo-visco-elasticity couples the equations for displacements with the heat equation. The solvability of the respective systems is however far from easy (see for instance [160] and [165]). Sometimes it is possible to prove only the local-in-time existence of the solution as can be seen in [215]. The whole mathematical theory related to elasticity has also a big influence on the theory of more general nonlinear partial differential equations [178].

Now, we pay attention to the main ideas related to the dimension reduction. Let us start with the static linear elasticity represented by the equation

$$-\mathrm{div}\,(A(D\mathbf{u})) = \mathbf{f} \text{ in } Q, \tag{1.1.1}$$

where

$$(A)_{ijkl} = A^{ijkl} := \lambda\delta^{ij}\delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}). \tag{1.1.2}$$

λ and μ are the Lamé constants related to elastic response and δ^{ij} stands for the Kronecker delta and $D\mathbf{u}$ is the symmetric part of the gradient of a displacement, i.e. $D\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$. To ensure the existence and uniqueness of the solution to (1.1.1)–(1.1.2) we must add the boundary conditions. Concerning the rest of the thesis, we assume the boundary conditions

$$\mathbf{u}(x) = 0,\ x \in \Gamma_1,\ A(D\mathbf{u})\mathbf{n}(x) = \mathbf{h},\ x \in \Gamma_2, \tag{1.1.3}$$

where $\Gamma_1 \cup \Gamma_2 = \partial Q$ and Γ_1 and Γ_2 are of nonzero measures. Under the boundary conditions (1.1.3), it is possible to prove the existence and the uniqueness of the solution (see for instance [42] or [159]). The key ingredient of the proof is Korn's inequality which can have various forms. Two of its more known forms in three dimensions are

$$\|\mathbf{u}\|_{1,p} \le C(\|D\mathbf{u}\|_p + \|\mathbf{u}\|_p),\ \forall \mathbf{u} \in W^{1,p}(\Omega)^3,$$

and

$$\|\mathbf{u}\|_{1,p} \le C\|D\mathbf{u}\|_p,\ \forall \mathbf{u} \in W_0^{1,p}(\Omega)^3,$$

for $p \in (1,\infty)$ (see for instance [72] and the references therein).

Let us start with a dimension reduction overview. First, we start with straight domains to introduce the main ideas and then we continue with results on deformed domains. If we assume that (1.1.1)–(1.1.3) corresponds to the three-dimensional model of elasticity then we can try to reduce the model to two or one dimension. In the first case, we assume we have domain $\Omega_\epsilon := S \times (0,\epsilon)$ or $\Omega_\epsilon := S \times (-\epsilon,\epsilon)$ and, in the second case, we have $\Omega_\epsilon := (0,l) \times \epsilon S$, $\epsilon > 0$, $S \subset \mathbb{R}^2$. If $\epsilon \to 0$ we get $\Omega_\epsilon \to S$ or $\Omega_\epsilon \to (0,l)$, respectively. The question is what happens with the solutions of (1.1.1).

First, we pay attention to plates and the respective limit $\Omega_\epsilon \to S$. One approach, to deal with the plates, is to start with three-dimensional linearized elasticity and to make a priori assumptions of a geometrical and a mechanical nature. Another approach is to apply the limit for $\epsilon \to 0$. In the end, it is necessary to compare the results of both approaches. Let us start with the first approach. As a geometrical assumption, we can apply the Kirchhoff-Love hypothesis that assumes that the normals to the middle surface stay normal to the deformed middle surface and the distance of any point on these normals to the middle surface remains constant [217]. From the mechanical point of view, we can assume that the stress field is "planar", in the sense that $\sigma_{i3}^\epsilon = 0$ [109]. An alternative approach is based on the so-called hierarchic theories that assume that the unknown displacements and stresses depend explicitly on the thickness coordinate. We refer the reader to [152] for the detailed explanation. Let us also mention the approach based on integrating the three-dimensional equations across the thickness followed by approximating the resulting equation by quadrature formulas [206]. It is important that using the asymptotic methods we give a complete mathematical justification of the classical linear and nonlinear Kirchhoff-Love theory.

Now, we introduce more closely the asymptotic methods. As mentioned above there are three ways to study the behavior of the solutions of (1.1.1) for $Q = \Omega_\epsilon$. The first approach is based on the direct estimates of the difference between the solution of the three-dimensional model and the lower-dimensional model. In this case, we must know, however, what the lower-dimensional model looks like. We refer the reader to [151], [162], [189] and [103] for a historical overview. The second approach is based on the constrained method, whose main principle is an a priori assumption that the admissible displacement fields are restricted to a specific form. We refer the reader to [152], [143], [20] and [185]. We will pay the largest attention to the third approach that is closely related to the techniques used in the thesis.

In the third approach, we have several ways to treat the limit $\epsilon \to 0$. The first technique is based on formal asymptotic methods that were successfully applied for the first time in [84] and [85]. To the variational, or weak, formulation they were applied in [47]. The main aim is, however, to apply the asymptotic methods for the rigorous asymptotic analysis which shows the convergence of the solutions of the respective three-dimensional problems in suitable function spaces. For the more rigorous approach, we refer the reader to the papers [49], [102], [103], [27], [68], [62], [3] and [2]. It is important to point out that the proofs rely on the ideas and methods developed in [117]. The analysis of the behavior of the magnitudes of the components of the loads and of Lamé constants that enable to derive the lower-dimensional model was given in [144], [145], and [104]. The extension to von Kármán plates can be found in [41].

Let us now demonstrate the main ideas and problems in the asymptotic analysis. Let us assume we have the equation

$$-\bar{\operatorname{div}}\,(\bar{A}_\epsilon(\bar{D}\bar{\mathbf{u}}_\epsilon)) = \bar{\mathbf{f}}_\epsilon \text{ in } \Omega_\epsilon := S \times (0,\epsilon), \tag{1.1.4}$$

where

$$\bar{A}_\epsilon^{ijkl} := \lambda_\epsilon \delta^{ij}\delta^{kl} + \mu_\epsilon(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}). \tag{1.1.5}$$

The used notation with bars refers to the domain Ω_ϵ. The boundary conditions read as follows

$$\bar{\mathbf{u}}_\epsilon = 0 \text{ on } \partial S \times (0,\epsilon),\ \bar{A}_\epsilon(\bar{D}\bar{\mathbf{u}}_\epsilon)\bar{\mathbf{n}}_\epsilon = \bar{\mathbf{h}}_\epsilon \text{ on } S \times \{0,\epsilon\}. \tag{1.1.6}$$

In view of (1.1.4)–(1.1.6), we can arrive at the weak formulation

$$\int_{\Omega_\epsilon} \bar{A}_\epsilon^{ijkl} \bar{D}_{kl}\bar{\mathbf{u}}_\epsilon \bar{D}_{ij}\bar{\mathbf{v}}_\epsilon \, dx_\epsilon = \int_{\Omega_\epsilon} \bar{\mathbf{f}}_\epsilon \cdot \bar{\mathbf{v}}_\epsilon \, dx_\epsilon + \int_{S\times\{0,\epsilon\}} \bar{\mathbf{h}}_\epsilon \cdot \bar{\mathbf{v}}_\epsilon \, dS_\epsilon. \tag{1.1.7}$$

The solution of (1.1.7) and the test functions are from the space

$$V(\Omega_\epsilon) := \{\bar{\mathbf{v}}_\epsilon :\ \bar{\mathbf{v}}_\epsilon \in W^{1,2}(\Omega_\epsilon)^3,\ \bar{\mathbf{v}}_\epsilon = 0 \text{ on } \partial S \times (0,\epsilon)\}.$$

We can now proceed with the following steps (see [43]).

1. We transform (1.1.7) on the referential domain $\Omega := S \times (0,1)$ that is ϵ-independent. The problem is that using the Chain rule we have

   $$\bar{\partial}_\alpha = \partial_\alpha,\ \alpha = 1,2,\ \bar{\partial}_3 = \frac{1}{\epsilon}\partial_3,$$

   which leads to the asymmetric terms in the symmetric part of the gradient, i.e. some of the terms are multiplied by $\frac{1}{\epsilon}$.

2. We use the scaling for the solution and test functions in (1.1.7)

$$\bar{u}_{\alpha,\epsilon}(x_\epsilon) = \epsilon^2 u_{\alpha,\epsilon}(x),\ \alpha = 1,2,\ \text{and}\ \bar{u}_{3,\epsilon}(x_\epsilon) = \epsilon u_{3,\epsilon}(x),$$

where $\pi^\epsilon : \Omega \to \Omega_\epsilon$ and $x_\epsilon = \pi^\epsilon(x)$.

3. We use the scaling for body force density and surface force density

$$\bar{f}_{\alpha,\epsilon}(x_\epsilon) = \epsilon^2 f_\alpha(x),\ \alpha = 1,2,\ \text{and}\ \bar{f}_{3,\epsilon}(x_\epsilon) = \epsilon^3 f_3(x),$$

$$\bar{h}_{\alpha,\epsilon}(x_\epsilon) = \epsilon^3 h_\alpha(x),\ \alpha = 1,2,\ \text{and}\ \bar{h}_{3,\epsilon}(x_\epsilon) = \epsilon^4 h_3(x).$$

The idea of scaling is to prevent the solutions $\mathbf{u}_\epsilon$ from spinning out of control for $\epsilon \to 0$.

Despite the above-mentioned transformation and scalings, there remain two main problems.

1. How to use Korn's inequality because its constant is domain-dependent.

2. After the limit process, unknown quantities related to the third coordinate appear and they must be expressed or eliminated from the limit equations.

How to solve the problems will be clear from the thesis.

The above-mentioned scalings are not a unique way to scale or treat (1.1.7). The older approach in [47] is based on the formal power series expansion of the solution and involves the following steps.

1. The first step is again the transformation of (1.1.7) on the referential domain $\Omega := S \times (0,1)$ that is ϵ-independent and the application of a suitable scaling for the displacement and the stress tensor.

2. The second step is to express the solution $\mathbf{u}_\epsilon$ as

$$\mathbf{u}_\epsilon = \frac{1}{\epsilon^2}\mathbf{u}_{-2} + \frac{1}{\epsilon}\mathbf{u}_{-1} + \mathbf{u}_0 + \epsilon\mathbf{u}_1 + \dots$$

3. We derive relations for $\mathbf{u}_i$, $i = -2, -1, \dots$, from (1.1.7) after the transformation on the referential domain Ω.

We will not pay much attention to the formal power series expansion because we do not work with it in the thesis but it is good to know about its existence.

Let us now go back to the first approach from [43]. Let us assume that the Lamé constants are independent of ϵ. The main result can be summarized as follows:

1.
$$\mathbf{u}_\epsilon \to \mathbf{u} \text{ in } W^{1,2}(\Omega)^3;$$

2. the limit function $\mathbf{u}$ satisfies

$$\int_\Omega \left[\frac{2\lambda\mu}{\lambda+2\mu} D_{\sigma\sigma}\mathbf{u} D_{\tau\tau}\mathbf{v} + 2\mu D_{\alpha\beta}\mathbf{u} D_{\alpha\beta}\mathbf{v}\right]\, dx = \int_\Omega \mathbf{f}\cdot\mathbf{v}\, dx + \int_{S\times\{0,1\}} \mathbf{h}\cdot\mathbf{v}\, dS,$$

where

$$\mathbf{u},\ \mathbf{v} \in \{\mathbf{w} \in W^{1,2}(\Omega)^3,\ \mathbf{w} = 0 \text{ on } \partial S \times (0,1),\ D_{i3}\mathbf{w} = 0\}$$

for $i = 1,2,3$ and $\sigma,\ \tau,\ \alpha,\ \beta = 1,2$. We also use the Einstein summation convention.

Even though the limit model seems to be three-dimensional, it is possible to prove, after a careful analysis, that the solution $\mathbf{u}$ is determined by the function ζ solving a two-dimensional problem from the Kirchhoff-Love theory of plates. The whole process can also be studied under various kinds of lateral boundary conditions [63] and [64]. Let us point out that the proof of the results is not based on the formal power series expansion and is the rigorous derivation of the lower-dimensional equations.

One of the most important things is the error estimate obtained in [67] for the norm $\sum_{i,j} \|\bar{D}_{ij}(\bar{\mathbf{u}}_\epsilon) - \bar{D}_{ij}(\bar{\mathbf{u}}_{0,\epsilon})\|_{2,\Omega_\epsilon}$, where $\bar{\mathbf{u}}_\epsilon$ is the original three-dimensional displacement field for the linearized elasticity and $\bar{\mathbf{u}}_{0,\epsilon}$ is the displacement field found by the Kirchhoff-Love theory. Let us remind the result from [172] where the upper bounds of the difference between the exact three-dimensional solution and a solution computed by using the Kirchhoff-Love hypotheses were derived. See also [149] for another approach.

The main problem with linear models is their limited applicability. This is the reason why attention is also paid to nonlinear models. The advantage of the models is that they can describe reality more precisely and sometimes also improve the properties of the solution. On the other hand, the nonlinearity brings additional technical difficulties to the proofs. We now give a brief overview of the dimension reduction approach for nonlinear elasticity. It seems to be a bit surprising but the formal asymptotic method can be also used for nonlinear models [48]. It seems to be, however, more natural to use a suitable scaling followed by the analysis of resulting convergences. It is also possible to apply Γ-convergence theory as in [112].

Let us briefly show the main ideas of the formal asymptotic expansion in nonlinear plate theory. The model of nonlinear elasticity has the following form

$$-\bar{\partial}_j(\bar{\sigma}^\epsilon_{ij} + \bar{\sigma}^\epsilon_{kj}\bar{\partial}_k \bar{u}_{i,\epsilon}) = \bar{f}_{i,\epsilon} \text{ in } \Omega_\epsilon, \tag{1.1.8}$$

$$\bar{u}_{i,\epsilon} = 0 \text{ on } \partial S \times (0,\epsilon), \tag{1.1.9}$$

$$(\bar{\sigma}^\epsilon_{ij} + \bar{\sigma}^\epsilon_{kj}\bar{\partial}_k \bar{u}_{i,\epsilon})\bar{n}_{i,\epsilon} = 0 \text{ on } S \times \{0,\epsilon\}, \tag{1.1.10}$$

where

$$\bar{\sigma}^\epsilon_{ij} := \lambda_\epsilon \bar{E}^\epsilon_{pp}(\bar{\mathbf{u}}_\epsilon)\delta^{ij} + 2\mu_\epsilon \bar{E}^\epsilon_{ij}(\bar{\mathbf{u}}_\epsilon)$$

and

$$\bar{E}^\epsilon_{ij}(\bar{\mathbf{u}}_\epsilon) := \frac{1}{2}(\bar{\partial}_i \bar{u}_{j,\epsilon} + \bar{\partial}_j \bar{u}_{i,\epsilon} + \bar{\partial}_i \bar{u}_{m,\epsilon}\bar{\partial}_j \bar{u}_{m,\epsilon}).$$

We use again the Einstein summation convention. The first formal asymptotic method covers the asymptotic expansion of the displacement and consists of the following steps:

1. We transform (1.1.8)–(1.1.10) on the referential domain $\Omega := S \times (0,1)$ similarly as in the linear case.

2. We use the scalings

$$\bar{u}_{\alpha,\epsilon}(x_\epsilon) = \epsilon^2 u_{\alpha,\epsilon}(x),\ \alpha = 1,2, \text{ and } \bar{u}_{3,\epsilon}(x_\epsilon) = \epsilon u_{3,\epsilon}(x),$$

   and

$$\bar{f}_{\alpha,\epsilon}(x_\epsilon) = \epsilon^2 f_\alpha(x),\ \alpha = 1,2, \text{ and } \bar{f}_{3,\epsilon}(x_\epsilon) = \epsilon^3 f_3(x),$$

   for the displacement and the body force density, respectively. We further assume that the Lamé constants do not depend on ϵ.

3. We use the following expansion

$$\mathbf{u}_\epsilon = \mathbf{u}_0 + \epsilon \mathbf{u}_1 + \epsilon^2 \mathbf{u}_2 + \epsilon^3 \mathbf{u}_3 + \epsilon^4 \mathbf{u}_4 + \ldots.$$

4. All terms $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\mathbf{u}_4$ must be studied before the leading term $\mathbf{u}_0$ can be identified.

The results of the approach are similar to the results in the linear case and are related to the classical Kirchhoff-Love theory of a nonlinearly elastic clamped plate.

The second approach is based on scaling and the asymptotic expansions of the displacement and stress tensor and proceeds as follows

1. We transform (1.1.8)–(1.1.10) on the referential domain $\Omega := S \times (0,1)$ similarly as in the linear case.

2. We use the scalings

$$\bar{u}_{\alpha,\epsilon}(x_\epsilon) = \epsilon^2 u_{\alpha,\epsilon}(x),\ \alpha = 1,2,\ \text{and}\ \bar{u}_{3,\epsilon}(x_\epsilon) = \epsilon u_{3,\epsilon}(x),$$

$$\bar{f}_{\alpha,\epsilon}(x_\epsilon) = \epsilon^2 f_\alpha(x),\ \alpha = 1,2,\ \text{and}\ \bar{f}_{3,\epsilon}(x_\epsilon) = \epsilon^3 f_3(x),$$

$$\bar{\sigma}^\epsilon_{\alpha\beta}(x_\epsilon) = \epsilon^2 \sigma^\epsilon_{\alpha\beta}(x),\ \bar{\sigma}^\epsilon_{\alpha 3}(x_\epsilon) = \epsilon^3 \sigma^\epsilon_{\alpha 3}(x),\ \bar{\sigma}^\epsilon_{33}(x_\epsilon) = \epsilon^4 \sigma^\epsilon_{33}(x),$$

α, $\beta = 1, 2$, for the displacement, the body force density, and the stress tensor. We further assume that the Lamé constants do not depend on ϵ.

3. Denoting $\Sigma_\epsilon = (\sigma^\epsilon_{ij})$ we can employ the following expansions

$$\mathbf{u}_\epsilon = \mathbf{u}_0 + \epsilon \mathbf{u}_1 + \epsilon^2 \mathbf{u}_2 + \dots$$

and

$$\Sigma_\epsilon = \Sigma_0 + \epsilon \Sigma_1 + \epsilon^2 \Sigma_2 + \dots.$$

4. The key is to identify the leading terms $\mathbf{u}_0$ and Σ_0.

Even in this case, we can find the standard two-dimensional equations of the nonlinear Kirchhoff-Love plate theory. As it was claimed in [43], the nonlinear Kirchhoff-Love theory is a small displacement theory. It is valid if the transverse displacements remain of the order of the thickness of the plate (see the scaling for the components of the displacement above). In contrast, there is also the theory of large displacements for nonlinear elasticity that can be again justified by an asymptotic analysis as in [79] and [112]. Another generalization is related to the asymptotic analysis of plates with periodically rapidly varying heterogenities [170].

In the case of nonlinear elasticity, it is interesting that as a result of the application of the asymptotic method, we get a partial linearization of the three-dimensional equations, i.e. the system, which was originally quasilinear, starts to be semilinear after the limit passage $\epsilon \to 0$. Due to this linearization, it was possible to establish more satisfactory results for the two-dimensional nonlinear plate equations than for the three-dimensional ones.

It is also possible to adapt the method of formal asymptotic expansions to the time-dependent problems for nonlinearly elastic plates to justify the time-dependent nonlinear Kirchhoff-Love theory [171].

The natural generalization of the theory of plates is the theory of shells. The shells can be understood as deformed plates. The proofs of the existence and uniqueness of the solution to the respective three-dimensional models for linear elasticity are again based on Korn's inequality. There is no significant difference in the proofs between plates and shells because they are studied in Cartesian coordinates. To apply an asymptotic method it is, however, necessary to express the respective equations in curvilinear coordinates. The mapping defined in (4.5.1) is a composition of two mappings:

$$\Omega := S \times (0,1) \to \Omega_\epsilon := S \times (0,\epsilon) \to \widetilde{\Omega}_\epsilon$$

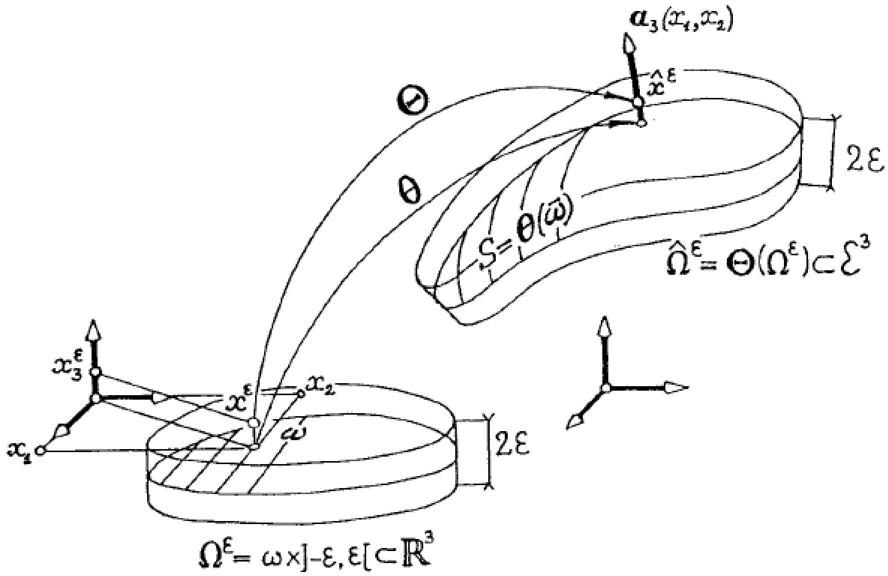

Figure 1: A shell and its curvilinear coordinates [44]

or

$$\Omega := S \times (-1,1) \to \Omega_\epsilon := S \times (-\epsilon, \epsilon) \to \widetilde{\Omega}_\epsilon.$$

The second mapping can be seen in Figure 1.

Let us now discuss the weak formulation of (1.1.1)–(1.1.3), its transformation to the curvilinear coordinates, and related problems. To be consistent with the rest of the habilitation thesis we use the notation with $\widetilde{\cdot}$ for the Cartesian coordinates, i.e. we put, for instance,

$$\widetilde{A}^{ijkl} := \lambda \delta^{ij} \delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}).$$

Using the standard process, we can derive the weak formulation of (1.1.1)–(1.1.3)

$$\int_{\widetilde{\Omega}_\epsilon} \widetilde{A}^{ijkl} \widetilde{D}_{kl} \widetilde{\mathbf{u}}_\epsilon \widetilde{D}_{ij} \widetilde{\mathbf{v}} \, d\widetilde{y} = \int_{\widetilde{\Omega}_\epsilon} \widetilde{\mathbf{f}}_\epsilon \cdot \widetilde{\mathbf{v}} \, d\widetilde{y} + \int_{\widetilde{S}_\epsilon} \widetilde{\mathbf{h}}_\epsilon \cdot \widetilde{\mathbf{v}} \, d\widetilde{S}_\epsilon \qquad (1.1.11)$$

if we assume that

$$\Gamma_1 := \Theta_\epsilon(S \times \{-1,1\}), \ \Gamma_2 = \widetilde{S}_\epsilon := \Theta_\epsilon(\partial S \times (-1,1)).$$

Let us now follow the introduction to the application of differential geometry to linearized elasticity from [44]. We show the main differences compared with plates. First, we neglect the parameter ϵ representing the thickness of the domain, i.e. we assume we have the proper mapping

$$\Theta : \Omega \to \widetilde{\Omega}$$

from the referential domain $\Omega := S \times (-1,1)$ to the deformed domain $\widetilde{\Omega}$. For more details, we refer the reader to Section 4.5. In case of $\widetilde{\Omega}$, we speak about the Cartesian coordinates but in the case of Ω, we speak about the curvilinear coordinates of $\widetilde{y} \in \widetilde{\Omega}$, i.e. we have $\widetilde{y} = \Theta(x), \ x \in \Omega$. The key components for the transformation of (1.1.11) to the referential domain Ω are the matrix

$$\nabla\Theta(x) = \begin{pmatrix} \partial_1\Theta_1 & \partial_2\Theta_1 & \partial_3\Theta_1 \\ \partial_1\Theta_2 & \partial_2\Theta_2 & \partial_3\Theta_2 \\ \partial_1\Theta_3 & \partial_2\Theta_3 & \partial_3\Theta_3 \end{pmatrix}(x) \qquad (1.1.12)$$

and the vectors of the covariant basis

$$\mathbf{g}_i(x) := \partial_i \Theta(x) = \begin{pmatrix} \partial_i \Theta_1 \\ \partial_i \Theta_2 \\ \partial_i \Theta_3 \end{pmatrix} (x). \tag{1.1.13}$$

It is natural to assume that we have the mapping Θ such that the vectors of the covariant basis are linearly independent. It is possible to check that each of the vectors $\mathbf{g}_i$ is tangent to the i-th coordinate line passing through $\widetilde{y} = \Theta(x)$, defined as the image by Θ of the points of $\widetilde{\Omega}$ that lie on the line parallel to the respective canonical basis vectors passing through x. We refer the reader to [44], Theorem 1.2-1 for the formulas how volume, area, and length elements at a point $\widetilde{y} = \Theta(x)$ can be expressed in terms of the matrix $\nabla\Theta$ or in terms of the covariant matrix $(\mathbf{g}_i \cdot \mathbf{g}_j)(x)$. If we define the vectors $\mathbf{g}^i$ of the contravariant basis as follows

$$\mathbf{g}^i \cdot \mathbf{g}_j = \delta^{ij},$$

we can get for any vector $\widetilde{\mathbf{u}}$ its standard or covariant components using the relations

$$u_j(x) = \widetilde{u}_i(\widetilde{y})[\mathbf{g}_j(x)]^i, \; \widetilde{u}_i(\widetilde{y}) = u_j(x)[\mathbf{g}^j(x)]_i, \; \widetilde{y} = \Theta(x).$$

Here we can see the main difference between the canonical basis and the covariant basis. The covariant components $u_i(x)$ represent the components of the displacement field over the basis $\{\mathbf{g}^1(x), \; \mathbf{g}^2(x), \; \mathbf{g}^3(x)\}$, which varies with $x \in \Omega$.

For the transformation of (1.1.11) with $\widetilde{\Omega}$ instead of $\widetilde{\Omega}_\epsilon$ to the referential domain Ω, it is necessary to use the Green formula together with the two relations

$$\widetilde{f}_i(\widetilde{y})\widetilde{v}_i(\widetilde{y}) = f^i(x)v_i(x), \; \widetilde{y} = \Theta(x), \; x \in \Omega, \tag{1.1.14}$$

where

$$v_i(x) = \widetilde{v}_j(\widetilde{y})[\mathbf{g}_i(x)]^j, \; f^i(x) = \widetilde{f}_j(\widetilde{y})[\mathbf{g}^i(x)]_j,$$

and

$$\widetilde{D}_{ij}\widetilde{\mathbf{v}}(\widetilde{y}) = \left(e_{k||l}(\mathbf{v})[\mathbf{g}^k]_i[\mathbf{g}^l]_j\right)(x), \tag{1.1.15}$$

where

$$e_{i||j}(\mathbf{v}) := \frac{1}{2}(\partial_j v_i + \partial_i v_j) - \Gamma^p_{ij} v_p. \tag{1.1.16}$$

The first relation is nothing but the invariance of the number $f^i(x)v_i(x)$ with concerning changes in curvilinear coordinates. In the second relation, we see the transformation of the symmetric part of the gradient. As a result of the transformation, the so-called Christoffel symbols appear, where

$$\Gamma^p_{ij} := \mathbf{g}^p \cdot \partial_i \mathbf{g}_j = \Gamma^p_{ji}. \tag{1.1.17}$$

The relation (1.1.16) can be understood as the generalization of the linearized strain tensor in the Cartesian coordinates to arbitrary curvilinear coordinates. After the transformation of (1.1.11) with $\widetilde{\Omega}$ instead of $\widetilde{\Omega}_\epsilon$ to the curvilinear coordinates, we arrive at

$$\int_\Omega A^{ijkl} e_{k||l}(\mathbf{u}) e_{i||j}(\mathbf{v}) \sqrt{g} \; dx = \int_\Omega f^i v_i \sqrt{g} \; dx + \int_{\partial S \times (-1,1)} h^i v_i \sqrt{g} \; dS, \tag{1.1.18}$$

where

$$A^{ijkl} := \lambda g^{ij} g^{kl} + \mu(g^{ik} g^{jl} + g^{il} g^{jk}), \; g^{ij} := \mathbf{g}^i \cdot \mathbf{g}^j, \; g := det(\mathbf{g}_i \cdot \mathbf{g}_j).$$

In (1.1.16) and (1.1.18) we can see the main problem with the transformation to the curvilinear coordinates, namely the presence of the Christoffel symbols that require

Θ to be at least C^2-diffeomorphism. Under the assumption, all functions remain in the spaces $W^{1,2}$ and L^2 after their transformation to the curvilinear coordinates. The next thing, that must be taken into account, is that Korn's inequality is still valid under the assumption. We refer the reader to [28], [197], [198], and [199] for the possible relaxation of the regularity assumptions and avoidance of the Christoffel symbols. We also refer the reader to [131] for the treatment of differential geometry and tensor analysis motivated by three-dimensional elasticity.

Let us assume we have Ω_ϵ instead of Ω in (1.1.18), i.e. we have

$$\int_{\Omega_\epsilon} \bar{A}_\epsilon^{ijkl} \bar{e}^\epsilon_{k||l}(\bar{\mathbf{u}}_\epsilon) \bar{e}^\epsilon_{i||j}(\bar{\mathbf{v}}_\epsilon)\sqrt{\bar{g}_\epsilon}\, dx_\epsilon = \int_{\Omega_\epsilon} \bar{f}^i_\epsilon \bar{v}_{i,\epsilon}\sqrt{\bar{g}_\epsilon}\, dx_\epsilon + $$

$$+ \int_{\partial S\times(-\epsilon,\epsilon)} \bar{h}^i_\epsilon \bar{v}_{i,\epsilon}\sqrt{\bar{g}_\epsilon}\, dS_\epsilon, \tag{1.1.19}$$

where

$$\bar{A}_\epsilon^{ijkl} := \lambda_\epsilon \bar{g}^{ij}_\epsilon \bar{g}^{kl}_\epsilon + \mu_\epsilon(\bar{g}^{ik}_\epsilon \bar{g}^{jl}_\epsilon + \bar{g}^{il}_\epsilon \bar{g}^{jk}_\epsilon).$$

The weak formulation (1.1.19) can be transformed to the referential domain $\Omega = S \times (-1,1)$. Moreover, we assume that

$$\bar{h}^i_\epsilon(x_\epsilon) = \epsilon^{p+1} h^i(x),\ \lambda_\epsilon = \lambda,\ \mu_\epsilon = \mu,\ \bar{f}^i_\epsilon(x_\epsilon) = \epsilon^p f^i(x),\ x_\epsilon = \pi_\epsilon(x),\ x \in \Omega.$$

Using the assumptions, we arrive at

$$\int_{\Omega} A_\epsilon^{ijkl} e^\epsilon_{k||l}(\mathbf{u}_\epsilon) e^\epsilon_{i||j}(\mathbf{v}_\epsilon)\sqrt{g_\epsilon}\, dx = \epsilon^p \int_{\Omega} f^i v_{i,\epsilon}\sqrt{g_\epsilon}\, dx + $$

$$+\epsilon^p \int_{\partial S\times(-1,1)} h^i v_{i,\epsilon}\sqrt{g_\epsilon}\, dS, \tag{1.1.20}$$

where

$$A_\epsilon^{ijkl} := \lambda g^{ij}_\epsilon g^{kl}_\epsilon + \mu(g^{ik}_\epsilon g^{jl}_\epsilon + g^{il}_\epsilon g^{jk}_\epsilon),$$

$$e^\epsilon_{\alpha||\beta}(\mathbf{v}) = \frac{1}{2}\left(\partial_\beta v_\alpha + \partial_\alpha v_\beta\right) - \Gamma^p_{\alpha\beta,\epsilon} v_p,$$

$$e^\epsilon_{\alpha||3}(\mathbf{v}) = \frac{1}{2}\left(\frac{1}{\epsilon}\partial_3 v_\alpha + \partial_\alpha v_3\right) - \Gamma^p_{\alpha 3,\epsilon} v_p,$$

and

$$e^\epsilon_{3||3}(\mathbf{v}) = \frac{1}{\epsilon}\partial_3 v_3.$$

We can now use the method of formal asymptotic expansions, i.e.

$$\mathbf{u}_\epsilon = \mathbf{u}_0 + \epsilon \mathbf{u}_1 + \epsilon^2 \mathbf{u}_2 + \epsilon^3 \mathbf{u}_3 + \epsilon^4 \mathbf{u}_4 + \ldots,$$

and we can try to identify the leading term $\mathbf{u}_0$. We have two possibilities related to the power p in (1.1.20). We can put $p = 0$ or $p = 2$. In the first case, we get that $\mathbf{u}_0$ satisfies the two-dimensional variational problem of a linear elastic "membrane" shell. In the second case, the resulting equations correspond to the two-dimensional variational problem of a linear elastic "flexural" shell. The main difference is among the behavior of various terms in the formal asymptotic expansions of the scaled linearized strains $e^\epsilon_{i||j}$. The expansion is the consequence of the formal asymptotic expansion for $\mathbf{u}_\epsilon$ and the linearity of the problem. The approach was pioneered in [147] for isotropic and homogeneous materials and further developed in [34] for non-homogeneous and anisotropic materials. There are two interesting aspects related to the limit linearly elastic flexural shell. The third component of the solution in the respective weak formulation of the limit equations is more regular, namely, it belongs to $W^{2,2}(S)$ and the Lamé constant λ is replaced by $\frac{4\lambda\mu}{\lambda+2\mu}$. The derivation using the method of formal asymptotic expansions is, however, the formal one and must be justified by convergence analysis. During the convergence analysis, it is checked that the leading term $\mathbf{u}_0$ is the limit of $\mathbf{u}_\epsilon$ for $\epsilon \to 0$. We refer the reader to [44] for a thorough overview and discussions.

We can see from (1.1.12)–(1.1.16) that the curvilinear coordinates make the problem of shells more complicated. The presence of the covariant and contravariant basis together with the Christoffel symbols require suitable geometrical and mechanical preliminaries. The main overview can be found in [44] but we also refer the reader to [50], [51], [53] for the geometrical preliminaries and to [24] and [50] for the mechanical preliminaries. The mechanical preliminaries comprise the behavior of the three-dimensional elasticity tensor A_ϵ^{ijkl} for $\epsilon \to 0$ and the uniform positive definiteness of the scaled two-dimensional elasticity tensor of the shell. During the study of the respective quantities, we can see various regularity assumptions ranging from C^1 to C^3, which makes room for possible relaxation of the regularity assumptions.

Let us now introduce the main results based on convergence analysis. In case of the linearly elastic membrane shells, i.e. for $p = 0$ in (1.1.20), it is possible to prove that

$$u_{\alpha,\epsilon} \to u_\alpha \text{ in } W^{1,2}(\Omega),\ \alpha = 1, 2, \text{ and } u_{3,\epsilon} \to u_3 \text{ in } L^2(\Omega) \tag{1.1.21}$$

for $\epsilon \to 0$. Moreover, u_i, $i = 1, 2, 3$, are independent of the transverse variable x_3. The limit function $\mathbf{u}$ is such that its average over x_3 is a unique solution to the associated scaled two-dimensional variational problem of a linearly elastic elliptic membrane shell. It is also possible to derive the respective boundary value problem and regularity results. The key ingredient is again Korn's inequality. It has, unfortunately, the following form

$$\left(\sum_{i=1}^{3} \|v_i\|_{1,2}^2\right)^{1/2} \le \frac{C}{\epsilon}\left(\sum_{i,j=1}^{3} \|e_{i||j}^\epsilon(\mathbf{v})\|_2^2\right)^{1/2} \tag{1.1.22}$$

Fortunately, it is possible to remove the dependence on ϵ in the constant $\frac{C}{\epsilon}$ and to prove that

$$\left(\sum_{\alpha=1}^{2} \|v_\alpha\|_{1,2}^2 + \|v_3\|_2^2\right)^{1/2} \le C\left(\sum_{i,j=1}^{3} \|e_{i||j}^\epsilon(\mathbf{v})\|_2^2\right)^{1/2} \tag{1.1.23}$$

for any $\mathbf{v} \in V(\Omega) := \{\mathbf{v} :\ \mathbf{v} \in W^{1,2}(\Omega)^3,\ \mathbf{v} = 0 \text{ on } \partial S \times (-1, 1)\}$. The next important result is related to the error estimates. In [128], it was proved that under suitable regularity assumptions

$$\|\mathbf{u}_\epsilon - \mathbf{u}\|_{W^{1,2}(\Omega)\times W^{1,2}(\Omega)\times L^2(\Omega)} \le C\epsilon^{1/6}.$$

The whole process can be generalized to linearly elastic generalized membrane shells. In this case, we must, however, assume that the applied forces contribute in a special way to the variational problem. The reason is that Korn's inequality (1.1.23) is not available for the estimate of the right-hand side of (1.1.20) and thus we do not have any uniform estimate with respect to ϵ. We refer the reader to [44], [51] and [129] for more details.

Concerning linearly elastic flexural shells, we can derive similar results. We must use the different scaling with $p = 2$ (see (1.1.20)) and we must take into account that the linearly elastic flexural shells depend on the subset of the lateral face, where the shells are subjected to boundary conditions of place, and on the geometry of the middle surfaces. Under suitable regularity assumptions on Θ_ϵ (C^3-regularity), it is possible to prove that

$$u_{i,\epsilon} \to u_i \text{ in } W^{1,2}(\Omega) \tag{1.1.24}$$

for $\epsilon \to 0$, u_i, $i = 1, 2, 3$, are independent of the transverse variable x_3 and

$$\hat{\mathbf{u}} := \frac{1}{2} \int_{-1}^{1} \mathbf{u}\, dx_3$$

satisfies the scaled two-dimensional variational problem for a linearly elastic flexural shell. The solution is, moreover, unique. We also refer to [52] for justification of the Koiter shell equations. One of the possible extensions of the results is the study of a partially clamped linearly elastic shell [120]. It is also useful to understand the asymptotic behavior of the stresses in thin elastic shells [59]. We also refer to [18] for an asymptotic algorithm for the derivation of equations of thin elastic shells that covers a boundary value problem for the Navier system in a thin region.

There are also further various extensions of the results. The first one considers time-dependent linearly elastic membrane and flexural shells, see [218] and [219]. Another possible generalization is related to the models with variable thickness, see [179] and [30], and [88] for the static and dynamic case, respectively. For the case of viscoelastic shells we refer the reader to [114], [36], [37], [38], [39], and [40]. There are also new results related to the asymptotic expansions [187], where the displacement is expanded with respect to the thickness variable of the middle surface. Another possible extension is to incorporate a rigid foundation [174]. It is also possible to add an obstacle to the model as in [113], [54], [55], and [167]. The next step is to study the problems with or without friction [142], [166], [17], and [168].

As in the case of plates, it is important to have some estimates between the solutions of three-dimensional models and two-dimensional models. Let us point to the results in [122], where the authors established a relative error estimate for the scaled linearized deformation tensor between the Koiter model or the Naghdi model and the respective three-dimensional model. We also refer to [61] for the energy estimate between the solution of the three-dimensional Lame system on a thin clamped shell and a displacement reconstructed from the solution of the classical two-dimensional Koiter model. In [116] there were derived some error estimates between the approximate solution of the asymptotic two-dimensional models and the three-dimensional displacement vector field of a flexural or membrane shell. Another approach based on formal series solutions can be found in [76]. Another error estimate between the solution of the Koiter model and the solution of a two-dimensional membrane shell problem is given in [130]. We also refer to [121] for the estimate of the difference between the solutions of the three-dimensional model and the two-dimensional Naghdi model for a thin shell. Except for the above-mentioned estimates, it is necessary to understand peculiarities that can appear when computing thin elastic shells [184].

The main difference between the asymptotic analysis of plates and shells is that in the case of plates, more freedom is allowed regarding assumptions and scalings of the displacement field and applied forces. Moreover, the limit problem of a linearly elastic plate includes at the same time flexural and membrane equations.

It is also possible to get similar results for nonlinear elasticity. Using the same scalings together with the methods of the formal asymptotic expansion, we can justify the two-dimensional model of a nonlinearly elastic membrane and flexural shell [146] and [123]. As in the linearized case, the leading term $\mathbf{u}_0$ is independent of the transverse variable and satisfies the limit equations. In addition, we must assume that the terms $\mathbf{u}_q$, $q = 0, 1, 2, 3, 4$, belong to $W^{1,4}(\Omega)^3$. We also refer to [56] for the justification of a two-dimensional nonlinear shell model of Koiter's type. In contrast to the linear problems, there are some unpleasant properties of the associated energy functional that is coercive but not sequentially weakly lower semicontinuous [81]. We also refer to [58], where the two-dimensional membrane models are obtained from the three-dimensional nonlinear models of a thin elastic shell made with a Saint Venant-Kirchhoff material. The next extension is related to a nonlinearly elastic membrane with periodically rapidly varying heterogeneities [169].

It is also interesting to study the relation between plates and shells. There are two possibilities for how to get plates from shells. The first one, the middle surface S converges to a planar domain and then $\epsilon \to 0$. In this case, we get the two-dimensional plate equations based on the Kirchhoff-Love theory. The second possibility is to change the order of the convergences. In this case, the limit is the membrane or flexural plate equation. It means that the two convergences do not commute (see [45], [46] and [183]).

The next interesting thing is the relation between the three-dimensional models of linear elasticity and two-dimensional linear shell theories. One of the most important of them is represented by the Koiter equations. The derivation of the equations is based on the observation that if the thickness is small enough the state of stress is approximately planar and the stresses parallel to the middle surface vary approximately linearly across the thickness [97] and [98]. If we take the approximations as an a priori mechanical assumption and if we combine them with the Kirchhoff-Love assumptions representing the geometrical assumptions, we can derive the Koiter equations [105], [106], and [107]. It means that the displacement field across the thickness of the shell is completely determined by the behavior of the displacement field in the points of the middle surface. The main result related to the above-discussed theory of shells is that the solution to the Koiter equations and the average of the solution of the three-dimensional problem over the third variable have the same limit for $\epsilon \to 0$ [66], [182], and [33]. The results are valid for an elliptic membrane shell, a generalized membrane shell, and a flexural shell. We also refer the reader to [186] for a new approximate model of a nonlinearly elastic flexural shell derived under the assumption that flexural energy is dominant. The paper also covers its numerical computation.

To study the similar problems related to the slender straight rods $\Omega_\epsilon := (0,l) \times \epsilon S$, it seems natural to exploit similar ideas as for plates. We can thus use again the constraint method (see for instance [14], [148], [141], and [16]) but the main technique remains the application of an asymptotic method. As a starting point, we can recommend to the reader the books [110] and [202]. As we know from the theory of plates, one of the asymptotic methods is based on formal asymptotic expansions. In the case of rods, we refer to [173], which seems to be the first attempt in that direction. The most important is, however, the application of a rigorous asymptotic analysis to the weak formulation of the elasticity problem, which involves the solution convergences in respective spaces as $W^{1,2}(\Omega)^3$ and $L^2(\Omega)^3$. The topic was pioneered in [1] for beams. The inspiration for the paper was results in [48]. It is interesting that in the paper the asymptotic expansion method is mentioned

$$(\mathbf{u}_\epsilon, \omega^\epsilon) = (\mathbf{u}_0, \omega_0) + \epsilon^2(\mathbf{u}_2, \omega_2) + \cdots \tag{1.1.25}$$

but not used in the proof. Instead of that, the authors use the classical convergence analysis based on the scaling

$$J_\epsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix}, \tag{1.1.26}$$

$$\mathbf{u}_\epsilon = J_\epsilon \hat{\mathbf{u}}_\epsilon,\ \mathbf{f}_\epsilon = J_\epsilon^{-1}\hat{\mathbf{f}}_\epsilon,\ \mathbf{h}_\epsilon = \frac{1}{\epsilon} J_\epsilon^{-1}\hat{\mathbf{h}}_\epsilon,\ \omega^\epsilon = J_\epsilon^{-1}\hat{\omega}_\epsilon J_\epsilon^{-1}, \tag{1.1.27}$$

where the quantities with hats represent the quantities as the displacement, external load, surface load, and stress tensor on the referential domain $\Omega := (0,l) \times S$. We also refer to [111].

As we can see in the case of plates and shells, the asymptotic expansion method can provide us with useful insight into the problem [78]. It is possible to give a complete characterization of displacements, bending moments, and shear forces of orders 0, 1, and 2 for linear elastic beams [201]. We also refer to [200] for the derivation of generalized models for linear elastic beams using the asymptotic expansion methods. The methods can be also applied to viscoelastic beam models [175]. One of the possible generalizations is to study beams with a variable cross-section [207] or anisotropic rods [4]. In the relation (4.2.1), we can see that some kind of symmetry must be required. However, it is possible to relax the assumption for the thin rods that are anisotropic, nonhomogeneous, and have periodic structure [155]. There were also proposed new models for variable cross-section rods in both symmetric and nonsymmetric cases using asymptotic methods in [8] and [11]. It is also possible to study the problem under a more specific external load [211]. The next application is on the model covering large deformations of a viscoelastic thin rod [25], where a Cosserat-based three-dimensional to one-dimensional reduction was studied. Another important topic is the influence of boundary conditions on the asymptotic analysis and the limit models. In [21] the authors studied a linear elasticity boundary value problem under the Robin boundary conditions at an end and on a segment of the lateral boundary in the middle of the beam, see also [35] for another kind of boundary conditions. One of the most important things for possible numerical applications is an error estimate between the solutions of three-dimensional models and the solution of the limit one-dimensional model [93].

In the case of plates, the natural generalization was to take shells. The same situation is in the case of rods, where we can take curved rods instead of straight rods or beams. We do not pay so much attention to the case now because it will be more properly discussed in Section 4. For linearly curved rods, we refer the reader to [96], [6], [95], [99], [153] and [100], and for shallow arches to [7], where the shallow arch is the kind of the curved rod where the curvature is of the order of the diameter of the cross-section. The main ideas and techniques of the convergence analysis are similar to the convergence analysis of plates and shells. One of the important topics for the curved rods is the regularity of their parametrizations, i.e. the regularity of the mapping

$$\bar{\mathbf{P}}_\epsilon : \Omega_\epsilon \to \widetilde{\Omega}_\epsilon. \tag{1.1.28}$$

The regularity of the mapping is, however, closely related to the regularity of the middle curve $\mathcal{C}$ of the curved rod $\widetilde{\Omega}_\epsilon$ that is given by its natural parametrization

$$\mathbf{\Phi} : [0,l] \to \mathbb{R}^3. \tag{1.1.29}$$

Let us start with [96] and [99], where $\mathbf{\Phi}$ was assumed to be of class C^4 and was used to define such curves which are called generic or biregular. The authors used the Frenet basis that requires the second derivative of $\mathbf{\Phi}$ to introduce the tangent, normal, and binormal vectors. For the definition of the covariant basis, it was then necessary to introduce the torsion that requires the third derivative of $\mathbf{\Phi}$. Similarly as in (1.1.16) the authors got the Christoffel symbols during the transformation of the equations for linear elasticity on a referential domain, which leads to the fourth derivative of $\mathbf{\Phi}$. As you can see the required regularity is very high and its relaxation is a natural step. Moreover, there are many practical situations in which the central line of the rod is not a generic curve and thus it can have vanishing curvature. The above-mentioned assumptions were relaxed in [100] for $\mathbf{\Phi} \in C^3([0,l])^3$, where the local right orthonormal basis was constructed. The regularity topic was further discussed in [196] for piecewise C^1 parametrizations, under the absence of surface tractions, and directly for the ordinary differential equations obtained as the asymptotic model in the smooth case. If we want to assume less regularity for $\mathbf{\Phi}$ the local frame $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{b}$ cannot be the Frenet one. We refer the reader to [92] for the construction of the Darboux frame under the C^1-regularity of $\mathbf{\Phi}$. It is also necessary to tackle the regularity problems related to the presence of the Christoffel symbols. It is again possible to use the idea developed originally for shells in [28] to remove the Christoffel symbols from the transformation on a referential domain (see Section 5.1.1 for more details). The next step is to assume that not only the thickness but also the shape of the curved rods depend on ϵ, which enables us to further relax the regularity assumptions on the limit of the sequence of middle curves. We refer to [199] for the detailed construction of the approximation scheme. The above-mentioned techniques of regularity relaxation can also be applied to time-dependent equations of linear elasticity [195] and [210].

At the end of the subsection, we give a brief overview related to the one-dimensional modeling of nonlinearly elastic rods and various applications. Even in the case of the nonlinearly elastic rods, we can use the same main steps:

1. Transformation on the referential domain $\Omega = (0, l) \times S$ together with suitable scaling.

2. Using the convergence analysis to derive a one-dimensional model. The convergence analysis need not be necessarily based on a formal asymptotic expansion.

We refer the reader to [57] for the first approach based on a formal asymptotic expansion. As we saw above, even this technique requires suitable scaling. In [57], it was necessary to scale the displacement, test functions, external and surface load, and the stress tensor. The formal asymptotic expansion was applied to the displacement and the stress tensor. In this case, the leading term of the formal asymptotic expansion of the scaled displacement field is a Bernoulli-Navier displacement field that satisfies a nonlinear ordinary differential equation of the fourth-order along the center line of the rod. We refer the reader to [202] and [150] for more details. It is also possible to extend the results to evolution models for nonlinearly elastic beams [10], see also [15] for the survey of some results. An example of another extension is the so-called genuinely clamped beam, that is the beam that is not only clamped at both ends but also at a neighborhood of them [9]. We also refer the reader to [19] for an alternative approach.

The technique of the asymptotic analysis or the dimension reduction can be extended to various problems related to elasticity. It is possible to study a thermoelastic model of rods [5] or various kinds of heat-conducting nonlinearly elastic curved rods [215]. Another natural extension concerns the asymptotic behavior of eigenvalues and eigenfunctions. The original problem for plates (see [49]) can also be studied for straight rods [101] and curved rods [200]. We also refer to [86] and [87] for an alternative approach based on the unfolding method. The method employs the fact that any displacement of a structure is the sum of an elementary displacement concerning the rods' cross-sections and a residual one related to the deformation of the cross-section. It is also possible to include friction to the models [94] and [208]. Asymptotic expansion methods can be also applied to an elastic rod in adhesive contact with a deformable foundation [177].

1.2 Fluids

As we can see in the previous section about elasticity, the asymptotic analysis of the elastic materials and the respective equations was not only of mathematical interest but also of practical interest, because it can be applied to various models and elevate the burden of numerical computation. It is thus natural to try to apply the main techniques and ideas to other materials. The most natural extension is to fluid mechanics because it is another large area of various models. It is thus no coincidence that one of the first results was published by Nazarov in [154] who also paid attention to elasticity [155]. He studied the Navier-Stokes system

$$-\triangle \mathbf{u}_\epsilon + (\mathbf{u}_\epsilon \cdot \nabla)\mathbf{u}_\epsilon + \nabla p_\epsilon = 0, \tag{1.2.1}$$

$$\operatorname{div} \mathbf{u}_\epsilon = 0 \tag{1.2.2}$$

in the domain $Q_\epsilon := \{x \in \mathbb{R}^3 : \ y = (x_1, x_2) \in S, \ -\epsilon h_1(y) \leq x_3 \leq \epsilon h_2(y)\}$, where h_1 and h_2 are smooth functions. The system was completed with the Dirichlet boundary conditions on the lower base and the lateral surface. On the upper base, there were either the Dirichlet conditions or free surface conditions. In the case of fluids, we omit for simplicity the notation with $\bar{\cdot}$. The same model was studied in [156], where a two-dimensional Reynolds-type equation was derived assuming viscous incompressible flow taking place between two smooth fixed adjacent curved walls under intensive percolation. We refer the reader to [82] for the derivation of the Reynolds equation governing the steady flow of a fluid through a curvilinear, narrow tube. The asymptotic expansion can also be used for the derivation of the second-order model correcting the standard Reynolds equation [137], [138] and [135]. The same technique can be applied to the same system but now in thin cylinders [164]. Naturally, the technique of the asymptotic expansion can also be applied to the Stokes system

$$-\triangle \mathbf{u}_\epsilon + \nabla p_\epsilon = 0, \tag{1.2.3}$$

$$\operatorname{div} \mathbf{u}_\epsilon = 0 \tag{1.2.4}$$

in thin cylinders [26]. The problem was also studied with the prescribed pressures at the pipe's ends [136]. We could see that the technique of asymptotic expansion is not the only one. In [213], the author studied to equations (1.2.1)–(1.2.2) in deformed domains using the description of the domain from Section 4.5 and the convergence analysis based on a priori estimates. It is also possible to look for inspiration in the homogenization theory as well [133].

The Navier-Stokes system (1.2.1)–(1.2.2) can also be generalized using shear-dependent viscosity to describe an incompressible viscous quasi-Newtonian fluid in a curved pipe with a smooth central curve [132]. Another application to the non-Newtonian flow in a thin domain between a rotating shaft and lubricated support can be seen in [74], where the incompressible Navier-Stokes (Stokes) system with a nonlinear viscosity was assumed. In this case, the Navier-Stokes system has the following form

$$-\operatorname{div} (|D\mathbf{u}_\epsilon|^{r-2} D\mathbf{u}_\epsilon) + (\mathbf{u}_\epsilon \cdot \nabla)\mathbf{u}_\epsilon + \nabla p_\epsilon = 0, \tag{1.2.5}$$

$$\operatorname{div} \mathbf{u}_\epsilon = 0 \tag{1.2.6}$$

with $r > 2$, where D is the symmetric part of the gradient. In [180], the authors studied the behavior of Newtonian and non-Newtonian fluids in a thin three-dimensional domain with mixed boundary conditions. It is also possible to assume that we have the irregular bottom of the domain with the presence of slight roughness of a given amplitude and period [192]. Another approach to the lubrication process [134] is via the Stokes system with the pressure-dependent viscosity, i.e.

$$-\operatorname{div} (\mu(p_\epsilon) D\mathbf{u}_\epsilon) + (\mathbf{u}_\epsilon \cdot \nabla)\mathbf{u}_\epsilon + \nabla p_\epsilon = 0, \tag{1.2.7}$$

$$\operatorname{div} \mathbf{u}_\epsilon = 0 \tag{1.2.8}$$

in $\Omega_\epsilon := \{x = (x', x_3) \in \mathbb{R}^3 : \ x' \in S, \ 0 < x_3 < \epsilon h(x')\}$, where

$$\mu(p) \sim e^{\alpha p}.$$

We also refer the reader to [139] and [140] for further extensions of the applications to the lubrication process. An interesting result can be found in [176], where the authors formally justified two models, a lubrication model and a shallow water model, using an asymptotic expansion method applied to the nonsteady Navier-Stokes equations for incompressible Newtonian fluids

$$\rho_0(\partial_t \mathbf{u}_\epsilon + (\mathbf{u}_\epsilon \cdot \nabla)\mathbf{u}_\epsilon) = -\nabla p_\epsilon + \mu \triangle \mathbf{u}_\epsilon + \rho_0 \mathbf{f}_\epsilon, \tag{1.2.9}$$

$$\operatorname{div} \mathbf{u}_\epsilon = 0. \tag{1.2.10}$$

It is also necessary to mention the paper [190], where the author provided the asymptotic analysis of the Oldroyd-type system

$$\partial_t \mathbf{u}_\epsilon + (\mathbf{u}_\epsilon \cdot \nabla)\mathbf{u}_\epsilon = -\nabla p_\epsilon + \mu \triangle \mathbf{u}_\epsilon + \operatorname{div}(FF^T), \tag{1.2.11}$$

$$\operatorname{div} \mathbf{u}_\epsilon = 0, \tag{1.2.12}$$

$$\partial_t F + \mathbf{u}_\epsilon \cdot \nabla F = \nabla \mathbf{u}_\epsilon F \tag{1.2.13}$$

in the rectangle $Q_\epsilon := (0,1) \times (0,\epsilon)$ with nontrivial velocity at the inflow and outflow area and the slip boundary condition at the rest of the boundary.

The next step is to study the limit problem for the inhomogeneous incompressible Navier–Stokes equations

$$\partial_t \rho_\epsilon + \operatorname{div}(\rho_\epsilon \mathbf{u}_\epsilon) = 0, \ \operatorname{div} \mathbf{u}_\epsilon = 0, \tag{1.2.14}$$

$$\partial_t(\rho_\epsilon \mathbf{u}_\epsilon) + \operatorname{div}(\rho_\epsilon \mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon) - \triangle \mathbf{u}_\epsilon + \nabla p_\epsilon = 0, \tag{1.2.15}$$

see [193].

In [212], the author studied the Navier-Stokes system for compressible, nonlinearly viscous fluids

$$\partial_t \rho_\epsilon + \operatorname{div}(\rho_\epsilon \mathbf{u}_\epsilon) = 0, \tag{1.2.16}$$

$$\partial_t(\rho_\epsilon \mathbf{u}_\epsilon) + \operatorname{div}(\rho_\epsilon \mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon) + \nabla p_\epsilon = \operatorname{div} \mathbb{S}_\epsilon + \rho_\epsilon \mathbf{f}_\epsilon \tag{1.2.17}$$

in the domain $\Omega_\epsilon := S \times (0,\epsilon)$. The stress tensor and pressure have the following forms

$$\mathbb{S}_\epsilon = P(|D\mathbf{u}_\epsilon|)D\mathbf{u}_\epsilon, \ p_\epsilon = \rho_\epsilon, \tag{1.2.18}$$

where the function P is of the exponential growth. The assumption seems to be a bit artificial but it is caused by the absence of the theory for P with the polynomial growth. The system (1.2.16)–(1.2.18) was coupled with the Navier boundary conditions and the respective two-dimensional model was derived. The used technique is not based on the asymptotic expansion but the energy inequality is employed for the derivation of a priori estimates. The result was further generalized to deformed domains in [13]. The same system was studied in thin cylinders in [12] as well. The most delicate problem, which was solved in the papers, was how to overcome nonlinearities $\rho_\epsilon \mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon$ and $P(|D\mathbf{u}_\epsilon|)D\mathbf{u}_\epsilon$.

We face similar problems with nonlinearities in the case of barotropic, compressible fluids, as well. The first attempt to apply the asymptotic analysis to the equations was published in [214] for the steady and nonsteady case. The nonsteady case is represented by the equations

$$\partial_t \rho_\epsilon + \operatorname{div}(\rho_\epsilon \mathbf{u}_\epsilon) = 0, \tag{1.2.19}$$

$$\partial_t(\rho_\epsilon \mathbf{u}_\epsilon) + \operatorname{div}(\rho_\epsilon \mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon) + \nabla \rho_\epsilon^\gamma = \operatorname{div}(2\mu D\mathbf{u}_\epsilon + \lambda \operatorname{div} \mathbf{u}_\epsilon I) + \rho_\epsilon \mathbf{f}_\epsilon + \mathbf{g}_\epsilon \tag{1.2.20}$$

in $\Omega_\epsilon = (0,l) \times \epsilon S$. The result obtained in [214] was, however, unsatisfactory because the author could not give a technique that would enable to overcome the nonlinearity in the pressure term ρ_ϵ^γ during the limit process. The term is very challenging and even in the case of the proof of the existence of a solution of (1.2.19)–(1.2.20), it was necessary to develop the ingenious technique that enables to overcome the nonlinearity [118] and [77]. In [22], the authors showed how to deal with the problems using the relative entropy inequality. The case with $\Omega_\epsilon := S \times (0, \epsilon)$ was treated in the same fashion in [124]. The problem with density-dependent viscosity was studied in [220]. The results were further extended to a flow of a general compressible viscous heat-conducting fluid in [29] in thin cylinders. The system has the governing equations

$$\partial_t \rho_\epsilon + \operatorname{div}(\rho_\epsilon \mathbf{u}_\epsilon) = 0, \tag{1.2.21}$$

$$\partial_t(\rho_\epsilon \mathbf{u}_\epsilon) + \operatorname{div}(\rho_\epsilon \mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon) + \nabla p(\rho_\epsilon, \theta_\epsilon) - \operatorname{div} \mathbb{S}(\theta_\epsilon, \nabla \mathbf{u}_\epsilon) = 0, \tag{1.2.22}$$

$$\partial_t(\rho_\epsilon s(\rho_\epsilon, \theta_\epsilon)) + \operatorname{div}(\rho_\epsilon s(\rho_\epsilon, \theta_\epsilon)\mathbf{u}_\epsilon) + \operatorname{div}\left(\frac{\mathbf{q}_\epsilon(\theta_\epsilon, \nabla\theta_\epsilon)}{\theta_\epsilon}\right) = \sigma_\epsilon, \tag{1.2.23}$$

where θ_ϵ is the temperature, $\mathbf{q}_\epsilon$ is the heat flux, σ_ϵ is the entropy production rate. Other applications are related to the asymptotic analysis of the motion of a viscous heat-conducting rotating fluid [70]. We also refer to [71], where the authors studied the three-dimensional compressible barotropic radiation fluid dynamics system describing the motion of the compressible rotating viscous fluid with gravitation and radiation confined to a straight layer $Q = S \times (0, \epsilon)$, where S is a two-dimensional domain. They showed that weak solutions to the respective three-dimensional system converge to the strong solution of the rotating two-dimensional Navier-Stokes-Poisson system. It is also possible to study the behavior of the solutions in generalized thin domains [115] or the compressible anisotropic Navier-Stokes equations

$$\partial_t \rho_\epsilon + \operatorname{div}(\rho_\epsilon \mathbf{u}_\epsilon) = 0 \tag{1.2.24}$$

$$\partial_t(\rho_\epsilon \mathbf{u}_\epsilon) + \operatorname{div}(\rho_\epsilon \mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon) + \nabla p(\rho_\epsilon) = \mu_{x'} \triangle_{x'} \mathbf{u}_\epsilon + \mu_{x_3} \partial^2_{x_3} \mathbf{u}_\epsilon \tag{1.2.25}$$

see [80]. The next approach is to assume that not only the thickness of the domain but also the Mach number tend to zero as in [31] and [32]. The above mentioned results are obtained under various kinds of boundary conditions such as the Dirichlet, slip and Navier boundary conditions but the periodic boundary conditions can be assumed as well [119].

2 Contribution to the theory

As we can see from the previous section, it is natural to distinguish between the respective problems in straight and curved domains. Even though the habilitation thesis covers only equations in the deformed domains (see Section 4), the author also made several contributions in the case of the straight domains. All of his results are summarized in the section.

2.1 Straight domains

In the case of straight domains, we paid attention only to the Navier-Stokes equations for the respective compressible fluids. The reason was that the topic seemed to be fresh and mostly unexplored for fluids. The first attempt was done in [212], where we studied the asymptotic behavior of the solutions to the Navier-Stokes equations for compressible, isothermal, and nonlinearly viscous fluids. The equations consist of

- **Continuity Equation**

$$\partial_t \bar{\rho}_\epsilon + \bar{\operatorname{div}}(\bar{\rho}_\epsilon \bar{\mathbf{u}}_\epsilon) = 0, \tag{2.1.1}$$

- **Momentum Equation**

$$\partial_t(\bar{\rho}_\epsilon \bar{\mathbf{u}}_\epsilon) + \bar{\operatorname{div}}\,(\bar{\rho}_\epsilon \bar{\mathbf{u}}_\epsilon \otimes \bar{\mathbf{u}}_\epsilon) + \bar{\nabla}\bar{p}_\epsilon = \bar{\operatorname{div}}\,\bar{\mathbb{S}}_\epsilon + \bar{\rho}_\epsilon \bar{\mathbf{f}}_\epsilon \tag{2.1.2}$$

in $\Omega_\epsilon := S \times (0, \epsilon)$, where $\bar{p}_\epsilon$ is the pressure, $\bar{\mathbb{S}}_\epsilon$ is the viscous stress tensor and $\bar{\mathbf{f}}_\epsilon$ stands for external forces. The solution is represented by the density $\bar{\rho}_\epsilon$ and the velocity $\bar{\mathbf{u}}_\epsilon = (\bar{u}_{1,\epsilon}, \bar{u}_{2,\epsilon}, \bar{u}_{3,\epsilon})$.

Since we asssume isothermal and nonlinearly viscous fluids, it means that

$$\bar{p}_\epsilon(\bar{\rho}_\epsilon) = c\bar{\rho}_\epsilon, \tag{2.1.3}$$

where we put $c = 1$, and

$$\bar{\mathbb{S}}_\epsilon = P(|\bar{D}\bar{\mathbf{u}}_\epsilon|)\bar{D}\bar{\mathbf{u}}_\epsilon, \tag{2.1.4}$$

where $\bar{D}$ is the symmetric part of the gradient. To ensure well-posedness of the problem, we must complete it with the Navier boundary conditions

$$\begin{gathered} \mathbf{t}_\epsilon \cdot (P(|\bar{D}\bar{\mathbf{u}}_\epsilon|)\bar{D}\bar{\mathbf{u}}_\epsilon \mathbf{n}_\epsilon) + q\bar{\mathbf{u}}_\epsilon \cdot \mathbf{t}_\epsilon = 0 \text{ on } \partial S \times (0, \epsilon) \times (0, T), \\ \mathbf{t}_\epsilon \cdot (P(|\bar{D}\bar{\mathbf{u}}_\epsilon|)\bar{D}\bar{\mathbf{u}}_\epsilon \mathbf{n}_\epsilon) + h(\epsilon)\bar{\mathbf{u}}_\epsilon \cdot \mathbf{t}_\epsilon = 0 \text{ on } [(S \times \{0\}) \cup (S \times \{\epsilon\})] \times (0, T), \\ \bar{\mathbf{u}}_\epsilon \cdot \mathbf{n}_\epsilon = 0 \text{ on } \partial\Omega_\epsilon \times (0, T), \end{gathered} \tag{2.1.5}$$

and with the initial conditions for the density and momentum:

$$\bar{\rho}_\epsilon(x, 0) = \bar{\rho}_{0,\epsilon}(x) \geq 0, \;\; (\bar{\rho}_\epsilon \bar{\mathbf{u}}_\epsilon)(x, 0) = (\bar{\rho}_\epsilon \bar{\mathbf{u}}_\epsilon)_0(x) \text{ in } \Omega_\epsilon. \tag{2.1.6}$$

Under suitable assumptions, we derived the respective two-dimensional model and we proved that the solutions to the three-dimensional models converge to a solution of the two-dimensional model. The main difficulties are the Orlicz and Sobolev-Orlicz spaces, where the solutions live, and the nonlinearity represented by the function P that is overcome by the monotonicity arguments.

The same system of equations was studied in [12] with $\Omega_\epsilon := (0, l) \times \epsilon S$. Using the asymptotic analysis we derived again the limit one-dimensional model and we proved the convergence of the solutions of the three-dimensional models to a solution of the one-dimensional model.

Another kind of fluid suitable for the asymptotic analysis contains barotropic, compressible fluids. The Navier-Stokes equations for the fluids and their solutions were studied in [214]. In this case, we have the following relations for the pressure and the stress tensor

$$\bar{p}_\epsilon(\bar{\rho}_\epsilon) = \bar{\rho}_\epsilon^\gamma \tag{2.1.7}$$

and

$$\bar{\mathbb{S}}_\epsilon = 2\mu\bar{D}\bar{\mathbf{u}}_\epsilon + \lambda\bar{\operatorname{div}}\,\bar{\mathbf{u}}_\epsilon I. \tag{2.1.8}$$

The problem was studied in the domains $\Omega_\epsilon := (0, l) \times \epsilon S$. The main task was to derive the limit one-dimensional model but the results were far from satisfactory because, as a result, we obtained only the limit triplet $(\rho, \mathbf{u}, \pi(\rho))$ solving the asymptotic one-dimensional equations, where $\pi(\rho)$ is a Radon measure representing a limit of the averages of pressure terms over the cross-sections of the channels. An analogous result was obtained for the steady version of the Navier-Stokes equations. The main difficulty, which is typical for this kind of equations, was how to overcome the nonlinearity in the pressure term.

2.2 Curved domains

In the case of curved or deformed domains, we have again two possibilities. The first one corresponds to curved rods and leads to one-dimensional models. The second one corresponds to shells and leads to two-dimensional models.

Let us start with an analogy of shells. The derivation of the two-dimensional limit model for the steady Navier-Stokes equations for incompressible flow can be found in [213]. The respective weak formulation of the equations looks like

$$\int_{\widetilde{\Omega}_\epsilon} [\widetilde{\nabla}\widetilde{\mathbf{u}}_\epsilon : \widetilde{\nabla}\widetilde{\psi}_\epsilon + (\widetilde{\mathbf{u}}_\epsilon \cdot \widetilde{\nabla})\widetilde{\mathbf{u}}_\epsilon \cdot \widetilde{\psi}_\epsilon - \widetilde{p}_\epsilon \widetilde{\operatorname{div}}\, \widetilde{\psi}_\epsilon]\, d\widetilde{y} +$$

$$+h(\epsilon)\int_{\Theta_\epsilon(S\times\{0\})\cup\Theta_\epsilon(S\times\{\epsilon\})} \widetilde{\mathbf{u}}_\epsilon \cdot \widetilde{\psi}_\epsilon\, d\widetilde{S} + q\int_{\Theta_\epsilon(\partial S\times(0,\epsilon))} \widetilde{\mathbf{u}}_\epsilon \cdot \widetilde{\psi}_\epsilon\, d\widetilde{S} =$$

$$= \int_{\widetilde{\Omega}_\epsilon} \widetilde{\mathbf{f}}_\epsilon \cdot \widetilde{\psi}_\epsilon\, d\widetilde{y}, \tag{2.2.1}$$

$$\widetilde{\operatorname{div}}\, \widetilde{\mathbf{u}}_\epsilon = 0 \text{ in } \widetilde{\Omega}_\epsilon \tag{2.2.2}$$

for $\widetilde{\psi}_\epsilon$ smooth enough and such that $\widetilde{\psi}_\epsilon \cdot \widetilde{\mathbf{n}}_\epsilon = 0$ on $\partial\widetilde{\Omega}_\epsilon$, which corresponds to the boundary conditions:

$$\widetilde{\mathbf{t}}_\epsilon \cdot (\widetilde{\nabla}\widetilde{\mathbf{u}}_\epsilon \widetilde{\mathbf{n}}_\epsilon) + q\widetilde{\mathbf{u}}_\epsilon \cdot \widetilde{\mathbf{t}}_\epsilon = 0 \text{ on } \Theta_\epsilon(\partial S \times (0,\epsilon)), \tag{2.2.3}$$

$$\widetilde{\mathbf{t}}_\epsilon \cdot (\widetilde{\nabla}\widetilde{\mathbf{u}}_\epsilon \widetilde{\mathbf{n}}_\epsilon) + h(\epsilon)\widetilde{\mathbf{u}}_\epsilon \cdot \widetilde{\mathbf{t}}_\epsilon = 0 \text{ on } \Theta_\epsilon(S \times \{0\}) \cup \Theta_\epsilon(S \times \{\epsilon\}), \tag{2.2.4}$$

$$\widetilde{\mathbf{u}}_\epsilon \cdot \widetilde{\mathbf{n}}_\epsilon = 0 \text{ on } \partial\widetilde{\Omega}_\epsilon, \tag{2.2.5}$$

where $q \geq 0$, $\widetilde{\mathbf{n}}_\epsilon$ is the unit outward normal to $\partial\widetilde{\Omega}_\epsilon$ and $\widetilde{\mathbf{t}}_\epsilon$ is any vector from the corresponding tangent plane. We refer the reader to Section 4.5 for more details about the definition of Θ_ϵ. After the transformation on a referential domain Ω, we proved that

$$\nabla\mathbf{u}_\epsilon \to \nabla\mathbf{u} \text{ in } W^{1,2}(\Omega)^3 \text{ and } \int_0^1 p_\epsilon\, dx_3 \to p \text{ in } L^2(S).$$

This kind of asymptotic analysis was also applied to the Navier-Stokes equations for isothermal, compressible, and nonlinearly viscous fluids (2.1.1)–(2.1.5) in curved domains (see [13]). However, in this case, it was impossible to prove the strong convergence of the velocity fields. The main obstacles were all nonlinear terms because it is necessary to work with the decomposition of the velocity field to the covariant and contravariant basis.

The second kind of curved domain is related to the curved rods. In this case, we published the paper [199], where we relaxed regularity assumptions on a parametrization of the Jordan unit speed curve for the limit one-dimensional model of linear elasticity. We also showed how to create smooth approximations of the parametrization for a three-dimensional model and how to use it in the asymptotic analysis. We proved that the solutions of the variational equations

$$\int_{\widetilde{\Omega}_\epsilon} \widetilde{A}^{ijkl}\widetilde{D}_{kl}\widetilde{\mathbf{u}}_\epsilon \widetilde{D}_{ij}\widetilde{\mathbf{v}}\, d\widetilde{y} = \int_{\widetilde{\Omega}_\epsilon} \widetilde{\mathbf{f}}_\epsilon \cdot \widetilde{\mathbf{v}}\, d\widetilde{y} + \int_{\widetilde{S}_\epsilon} \widetilde{\mathbf{g}}_\epsilon \cdot \widetilde{\mathbf{v}}\, d\widetilde{S}_\epsilon d\widetilde{y}_1 \tag{2.2.6}$$

converge strongly to a solution of the limit equation after their transformation on a referential domain. We were also able to express the limit forms of the components of the stress tensor. It is important to mention that the cross-section of the curved rods requires some kind of symmetry.

The equation (2.2.6) has two natural extensions. The first one covers its non-steady version. In this case, the limit equation and respective convergences were derived in [210]. We also refer the reader to Section 3 and 5 for the equation and the technique of the proof. The second extension covers the equation (2.2.6) completed with the special form of the body forces and the surface tractions, namely

$$\int_{\widetilde{\Omega}_\epsilon} \widetilde{A}^{ijkl}\widetilde{D}_{kl}(\widetilde{\mathbf{u}}_\epsilon)\widetilde{D}_{ij}(\widetilde{\mathbf{v}})\ d\widetilde{y} = \int_{\widetilde{\Omega}_\epsilon} \widetilde{\mathbf{f}}_\epsilon \cdot \widetilde{\mathbf{v}}\ d\widetilde{y} + \int_{\widetilde{\Omega}_\epsilon} \widetilde{H}_{ij,\epsilon}\widetilde{D}_{ij}(\widetilde{\mathbf{v}})\ d\widetilde{y} +$$

$$+ \int_{\widetilde{S}_\epsilon} \widetilde{\mathbf{g}}_\epsilon \cdot \widetilde{\mathbf{v}}\ d\widetilde{S}_\epsilon + \int_{\widetilde{S}_{l,\epsilon}} \widetilde{\mathbf{k}}_\epsilon \cdot \widetilde{\mathbf{v}}\ d\widetilde{S}_{l,\epsilon}. \tag{2.2.7}$$

Even in this case, we proved that the solutions of (2.2.7) converge strongly to a solution of the limit one-dimensional equation. It was also possible to express the limit form of the stress tensor. We refer the reader to [211] for more details. It is also possible to study more general cases as it was done in [215], where the dynamic, nonlinear model for heat conducting elastic materials was studied and its lower-dimensional version derived as a limit of the three-dimensional model. We refer the reader to equations (3.0.5)–(3.0.9) for more details.

3 Basic equations

In this section, we introduce three kinds of systems of partial differential equations we want to study in the next sections. Two of the systems correspond to elastic problems and were studied in papers [210] and [215]. The third system corresponds to the Navier-Stokes equations for compressible fluids that were studied in [13]. Except for the systems, we also suggest some difficulties in the proofs which must be overcome. The difficulties are related to the thinness of domains and nonlinear terms in respective equations.

The first equation comes from the theory of linear elasticity [42] and its time-dependent version can be expressed by the following equation

$$\rho\partial_{tt}\mathbf{u} - \operatorname{div}(A(D\mathbf{u})) = \mathbf{f} \text{ in } Q \times (0,T) \tag{3.0.1}$$

supplemented by the initial and boundary conditions:

$$\mathbf{u}(x,0) = \mathbf{u}_0(x),\ \partial_t\mathbf{u}(x,0) = \mathbf{u}_1(x),\ x \in Q, \tag{3.0.2}$$

$$\mathbf{u}(x,t) = 0,\ x \in \Gamma_1 \times (0,T),\ A(D\mathbf{u})\mathbf{n}(x,t) = \mathbf{h}(x,t),\ (x,t) \in \Gamma_2 \times (0,T), \tag{3.0.3}$$

where $\Gamma_1 \cup \Gamma_2 = \partial Q$ and the components of the operator A are given by

$$A^{ijkl} := \lambda\delta^{ij}\delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}). \tag{3.0.4}$$

$\mathbf{u}$ is a displacement, D is the symmetric part of the gradient, ρ is mass density and $\mathbf{n}$ is the unit outward normal to ∂Q.

Instead of the general domain Q, we assume we have a thin curved domain $\widetilde{\Omega}_\epsilon$ that can be defined as the image of a referential domain $\Omega := (0,l) \times S$ using mappings $\boldsymbol{R}_\epsilon$ and $\bar{\mathbf{P}}_\epsilon$ ((4.2.2) and (4.2.4)). We must also add symmetry assumptions (4.2.1). Using the definition we are able to transform the respective weak formulation from the domain $\widetilde{\Omega}_\epsilon$ that depends on thickness of the domain ϵ to a referential domain Ω that does not depend on ϵ.

As the first step, it is necessary to ensure a priori estimates. It is, however, impossible without an appropriate scaling of the loads, i.e. of the functions $\mathbf{f}_\epsilon$ and $\mathbf{h}_\epsilon$. In the real world, it would be also necessary to modify loads because they could lead to a breakup of the object. Another key ingredient for the a priori estimates is

Korn's inequality. Its problem is that the respective constant is domain-dependent (see [43], [44]). After the transformation on referential domain Ω, we must prove its special version

$$\|\mathbf{v}\|_{1,2} \leq \frac{C}{\epsilon}\|\omega^{\epsilon}(\mathbf{v})\|_{2},$$

where $\omega^{\epsilon}(\mathbf{v})$ corresponds to the symmetric part of the gradient after the transformation on a referential domain. On the one hand, the inequality enables us to derive a priori estimates. On the other hand, dependence on ϵ causes problems in the derivation of a limit. To derive a respective one-dimensional model we must prove that the limit displacements $\mathbf{u}$ are independent of x_2 and x_3. We must also prove that the unknown limit functions from $\omega^{\epsilon}(\mathbf{u}_{\epsilon})$ are either equal to zero or can be expelled from the limit equation using special test functions.

The second system is much more general and consists of two coupled equations for displacement and heat

$$\rho\partial_{tt}\mathbf{u} - \mathrm{div}\,[\mathrm{div}\,(\lambda\mathbf{u} + \lambda_v\partial_t\mathbf{u})\mathbb{I} + 2(\mu D\mathbf{u} + \mu_v D\partial_t\mathbf{u}) - \upsilon(3\lambda + 2\mu)\vartheta\mathbb{I}] = \mathbf{f}, \quad (3.0.5)$$

$$c\partial_t\vartheta + \vartheta\partial_t(\upsilon(3\lambda + 2\mu)\mathrm{div}\,\mathbf{u}) = \mathrm{div}\,(\kappa(\vartheta)\nabla\vartheta) + h \text{ in } Q \times (0,T) \quad (3.0.6)$$

supplemented by the initial and boundary conditions

$$\mathbf{u}(x,t) = 0,\ (x,t) \in \Gamma_1 \times (0,T),\ (D\mathbf{u})\mathbf{n}(x,t) = 0,\ (x,t) \in \Gamma_2 \times (0,T), \quad (3.0.7)$$

$$\Gamma_1 \cup \Gamma_2 = \partial Q,\ (\nabla\vartheta \cdot \mathbf{n})(x,t) = 0,\ (x,t) \in \partial Q \times (0,T), \quad (3.0.8)$$

where Γ_1 and Γ_2 will be specified later,

$$\mathbf{u}(x,0) = \mathbf{u}_0(x),\ \partial_t\mathbf{u}(x,0) = \mathbf{u}_1(x),\ \vartheta(x,0) = \vartheta_0(x),\ x \in Q. \quad (3.0.9)$$

In (3.0.5)–(3.0.9) we use the following notation

- $\mathbf{u} : Q \times (0,T) \to \mathbb{R}^3$ is displacement,
- $\vartheta : Q \times (0,T) \to \mathbb{R}$ is temperature,
- D stands for the symmetric part of the gradient,
- $\lambda \geq 0$ and $\mu > 0$ are Lamé constants related to elastic response,
- υ is the coefficient of thermal expansion,
- $\gamma > 0$ is a regularizing coefficient reflecting bending rigidity,
- $c > 0$ is heat capacity,
- $\kappa > 0$ is heat conduction function,
- $\rho > 0$ is mass density,
- $\lambda_v \geq 0$ and $\mu_v > 0$ are Lamé constants related to viscous response,
- $\mathbf{f} : Q \times (0,T) \to \mathbb{R}^3$ is an external force,
- $h : Q \times (0,T) \to \mathbb{R}$ is an internal heat source.

The system (3.0.5), (3.0.6) can be derived from a more general model introduced in [178] under assumptions that the displacements and their velocities are small. Thus the higher-order terms can be neglected. To derive suitable a priori estimates we must introduce suitable scaling of respective functions $\mathbf{f}_{\epsilon}$ and h_{ϵ}. Comparing equation (3.0.5) to (3.0.1) we can see that another term is present, namely $D\partial_t\mathbf{u}$, which can cause problems related to Korn's inequality. Fortunately, the term can be treated in a similar way as $D\mathbf{u}$ and it is thus enough to modify the technique developed for the first system. Since in (3.0.6) we have $\nabla\vartheta$ we do not face the same difficulties as with the displacement $\mathbf{u}$. On the other hand, there is the nonlinear term $\kappa(\vartheta)$ that must be overcome during the limit process to get reasonable limit equations. Again we must prove the limit functions $\mathbf{u}$ and ϑ are independent of variables x_2 and x_3.

The last system for compressible, nonlinearly viscous fluids is represented by the Navier-Stokes equations

- **Continuity Equation**
$$\partial_t\rho + \mathrm{div}\,(\rho\mathbf{u}) = 0, \tag{3.0.10}$$

- **Momentum Equation**
$$\partial_t(\rho\mathbf{u}) + \mathrm{div}\,(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p = \mathrm{div}\,\mathbb{S} + \rho\mathbf{f} \tag{3.0.11}$$

in $Q\times(0,T)$, where $\mathbf{u}$ is the velocity, ρ is the density, p is the pressure, $\mathbb{S}$ is the viscous stress tensor and $\mathbf{f}$ stands for external forces.

In this paper, we pay attention to isothermal gas, which means

$$p(\rho) = c\rho, \tag{3.0.12}$$

where we put $c = 1$. We also restrict ourselves only to non-Newtonian fluids, i.e.

$$\mathbb{S} = P(|D\mathbf{u}|)D\mathbf{u}, \tag{3.0.13}$$

where D is the symmetric part of the gradient and the function P will be specified later. We complete (3.0.10), (3.0.11) by the set of the Navier boundary conditions

$$(\mathbf{t}\cdot(P(|D\mathbf{u}|)D\mathbf{u}\,\mathbf{n}) + h\mathbf{u}\cdot\mathbf{t})(x,t) = 0,\ (x,t)\in\Gamma_1\times(0,T), \tag{3.0.14}$$
$$(\mathbf{t}\cdot(P(|D\mathbf{u}|)D\mathbf{u}\,\mathbf{n}) + q\mathbf{u}\cdot\mathbf{t})(x,t) = 0,\ (x,t)\in\Gamma_2\times(0,T), \tag{3.0.15}$$

where Γ_1 and Γ_2, $\Gamma_1\cup\Gamma_2 = \partial Q$, will be specified later, and

$$(\mathbf{u}\cdot\mathbf{n})(x,t) = 0,\ (x,t)\in\partial Q\times(0,T), \tag{3.0.16}$$

to ensure the well-posedness of the problem. $\mathbf{t}$ is a tangent vector and $\mathbf{n}$ is an outer normal vector to ∂Q. The initial conditions are

$$\rho(x,0) = \rho_0(x)\geq 0,\ (\rho\mathbf{u})(x,0) = (\rho\mathbf{u})_0(x),\ x\in Q. \tag{3.0.17}$$

4 Preliminaries

In this section, we introduce basic notation used throughout the thesis together with function spaces. We also introduce curved domains together with their basic properties. The curved domains are then used for dimension reductions to one or two dimensions.

4.1 Basic notation and function spaces

We denote by $\mathbb{R}^3$ the usual three dimensional Euclidean space with scalar product " $\cdot$ " and the Euclidean norm $|\cdot|$. For the scalar product of tensors and the tensor product we use notations " : " and " $\otimes$ ", respectively. By " $\cdot\times\cdot$ " we shall denote the vector product in $\mathbb{R}^3$ and by $\langle\cdot,\cdot\rangle$ any ordered pair. In the text, the symbol " $\times$ " is also used for the Cartesian product of two spaces and $|A|$ will also denote the Lebesgue measure of some measurable set A, without danger of confusion. The summation convention with respect to repeated indices will be also used, if not otherwise explicitly stated. We use for constants the symbols C or C_i, for $i \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Constant vectors will be denoted by $\mathbf{C}$ or $\mathbf{C}_i$ for $i \in \mathbb{N}_0$.

Let $Q \subset \mathbb{R}^3$ be a bounded Lipschitz domain, i.e. its boundary can be described by Lipschitz continuous functions. If we denote the boundary of Q by ∂Q, we can write $\partial Q \in \mathcal{C}^{0,1}$. In the same context we use notation $\partial Q \in \mathcal{C}^k$, $k \in \mathbb{N}$. We also use well-known Sobolev, Lebesgue and Bochner spaces. The symbols $W^{1,p}(Q)$, $W_0^{1,p}(Q)$ and $L^p(Q)$, respectively, thus denote (for $p \in [1, \infty]$) the standard Sobolev and Lebesgue spaces endowed with the norms $\| \cdot \|_{1,p}$ or $\| \cdot \|_p$. In case we will work with vector or tensor functions with components in the Sobolev and Lebesque spaces we will use the notation $W^{1,p}(Q)^m$ or $W^{1,p}(Q)^{m \times n}$. It holds for other spaces as well. $[W_0^{1,p}(Q)]'$ stands for the dual space to $W_0^{1,p}(Q)$. Other dual spaces will be denoted similarly. The notation $C^m(\overline{Q})$, with $m \in \mathbb{N}_0$, means the usual spaces of continuous functions whose derivatives up to the order m are continuous in $\overline{Q}$. We can analogously define the space $C^\infty(\overline{Q})$. By notation $C_0^\infty(Q)$ we mean the space of C^∞-functions with compact supports. The space $\mathcal{D}(Q)$ consists of smooth (C^∞-) and compactly supported functions endowed with the inductive limit topology. The respective dual space is $[\mathcal{D}(Q)]'$. The symbols $L^p(0, T; X)$, $p \in [1, \infty]$, and $C([0, T]; X)$, where X is a Banach space, stand for the Bochner spaces endowed with the norms

$$\|v\|_{L^p(0,T;X)} := \left(\int_0^T \|v(t)\|_X^p \, dt \right)^{1/p} \quad \text{and } \|v\|_{C([0,T];X)} := \max_{t \in [0,T]} \|v(t)\|_X.$$

From the definition, it is clear how to establish other Bochner spaces as $L^\infty(0, T; X)$, $W^{1,p}(0, T; X)$, $p \in [1, \infty]$, and $C^q([0, T]; X)$, $q \in \mathbb{N}$. In the thesis, we will also use function spaces that are somehow related to classical Lebesgue and Sobolev spaces. Since the spaces are closely related to a specific chapter we do not introduce them here.

A natural generalization of the Lebesque and Sobolev spaces are the so-called Orlicz and Sobolev-Orlicz spaces denoted by $L_\Phi(Q)$, $W^1 L_\Phi(Q)$ and $W_0^1 L_\Phi(Q)$. Since the function spaces are not so widely used, we pay them more attention. During their introduction, we follow [108].

The theory of the Orlicz spaces is based on the definition of the Young function

$$\Phi(z) := \int_0^z \varphi(s) \, ds, \; z \geq 0,$$

where

1. $\varphi(0) = 0$;
2. $\varphi(s) > 0$ for $s > 0$;
3. φ is right continuous at any point $s \geq 0$;
4. φ is nondecreasing on $[0, \infty)$;
5. $\lim_{s \to \infty} \varphi(s) = \infty$.

The main difference between the Lebesgue and Orlicz spaces is that the so-called Orlicz class $\widetilde{L}_\Phi(Q)$ defined as a set of functions satisfying

$$\int_Q \Phi(|u(x)|) \, dx < \infty$$

is not generally a function space. To define the Orlicz spaces we have to introduce the norm

$$\|u\|_\Phi := \sup_v \int_Q |u(x)v(x)| \; dx,$$

where $v \in \widetilde{L}_\Psi(Q)$ and $\int_Q \Phi(|v(x)|) \; dx \leq 1$. Ψ is a complementary function to Φ defined by

$$\psi(t) := \sup_{\varphi(s) \leq t} s, \; t \geq 0, \; \Psi(z) := \int_0^z \psi(s) \; ds.$$

But even in this case, we get two spaces instead of one. The first function space (denoted by $L_\Phi(Q)$) is defined as a set of functions satisfying $\|u\|_\Phi < \infty$ and the second space (denoted by $E_\Phi(Q)$) as the closure $\overline{B(Q)}^{\|\cdot\|_\Phi}$, where $B(Q)$ is a set of bounded functions. Unfortunately the Orlicz class and the two function spaces do not coincide, i.e.

$$E_\Phi(Q) \subseteq \widetilde{L}_\Phi(Q) \subseteq L_\Phi(Q).$$

The sets coincide only if the Young function satisfies the so-called Δ_2-condtion, i.e.

$$\Phi(2z) \leq k\Phi(z), \text{ for all } z \geq z_0 \geq 0 \text{ and some constant } k > 0. \tag{4.1.1}$$

The discrepancy between the Lebesque and the Orlicz spaces has an unpleasant consequence in weak convergence. In case of the Orlicz spaces a sequence $\{u_n\}_{n=1}^{+\infty} \subset L_\Phi(Q)$ converges E_Ψ-weakly to $u \in L_\Phi(Q)$, if

$$\lim_{n\to+\infty} \int_Q (u_n(x) - u(x))v(x) \; dx = 0, \quad \forall v \in E_\Psi(Q).$$

We write $u_n \overset{\Psi}{\rightharpoonup} u$. The weak-* convergence in $L_\Phi(Q)$ is equivalent to the E_Ψ-weak convergence. Moreover, the boundedness of $\{u_n\}_{n=1}^{+\infty}$ in $L_\Phi(Q)$ implies the existence of an E_Ψ- weakly convergent subsequence of $\{u_n\}_{n=1}^{+\infty}$. We face the same problems with the definition of the Sobolev-Orlicz spaces. We can thus denote by

$$W^k L_\Phi(Q) \text{ and } W^k E_\Phi(Q)$$

the spaces such that the corresponding functions and all their distributional derivatives up to the order k belong to $L_\Phi(Q)$ and $E_\Phi(Q)$, respectively. Sobolev-Orlicz spaces $W_0^1 L_\Phi(Q)$ are generalizations of Sobolev spaces $W_0^{1,p}(Q)$. By $[W_0^1 L_\Phi(Q)]'$ we denote their dual spaces. We refer the reader to [108] for more details about any of the above-mentioned function spaces.

There are also three main inequalities.

- **Hölder's inequality:** Let $u \in L_\Phi(Q)$ and $v \in L_\Psi(Q)$, where Φ, Ψ is a pair of the complementary Young functions. Then $uv \in L^1(Q)$ and

$$\int_Q |u(x)v(x)| \; dx \leq \|u\|_{L_\Phi(Q)} \|v\|_{L_\Psi(Q)}. \tag{4.1.2}$$

- **Young's inequality** Let a, $b \in \langle 0, +\infty)$ and Φ, Ψ be the complementary Young functions. It holds that

$$ab \leq \Phi(a) + \Psi(b). \tag{4.1.3}$$

- **Jensen's inequality** Let us assume that $\Lambda : \mathbb{R} \to \mathbb{R}$ is a convex function and $\alpha(x)$ is positive almost everywhere in $Q \subset \mathbb{R}^n$, $n \in \mathbb{N}$. Then

$$\Lambda\left(\frac{\int_Q \alpha(x)u(x) \; dx}{\int_Q \alpha(x) \; dx}\right) \leq \frac{\int_Q \alpha(x)\Lambda(u(x)) \; dx}{\int_Q \alpha(x) \; dx} \tag{4.1.4}$$

  for any non-negative function $u : Q \to \mathbb{R}$ supposing that all the integrals in (4.1.4) are meaningful.

4.2 Curved rods and related function spaces

In this section, we define curved domains using referential "straight" domains. The notation is then used in Section 4 for elasticity problems and corresponds to curved rods. We also introduce function spaces related to the curved domains.

Let $S \subset \mathbb{R}^2$ be a bounded simply connected domain of class $\mathcal{C}^1$ satisfying the "symmetry" condition

$$\int_S x_2 \; dx_2 dx_3 = \int_S x_3 \; dx_2 dx_3 = \int_S x_2 x_3 \; dx_2 dx_3 = 0. \tag{4.2.1}$$

In Section 4, we denote by $\Omega := (0,l) \times S$ and $\Omega_\epsilon := (0,l) \times \epsilon S$ open "cylinders" in $\mathbb{R}^3$, where $l > 0$ and $\epsilon > 0$ "small", are given. Domain Ω is called a referential domain. Domain Ω_ϵ represents the thin domain derived from Ω. It can be represented by mapping $\mathbf{R}_\epsilon$

$$\mathbf{R}_\epsilon : \Omega \to \Omega_\epsilon, \; \mathbf{R}_\epsilon(x_1, x_2, x_3) = (x_1, \epsilon x_2, \epsilon x_3). \tag{4.2.2}$$

If we assume $\epsilon \to 0$ then $\Omega_\epsilon \to (0,l)$. Using the limit we pass from a three-dimensional domain to a one-dimensional beam.

The elastic domain may not be necessarily straight but may be deformed. To describe the deformed domains we must introduce their suitable descriptions. Let $\mathcal{C}_\epsilon$ be a Jordan unit speed curve of length l in $\mathbb{R}^3$ defined by its parametrization $\mathbf{\Phi}_\epsilon : [0,l] \to \mathbb{R}^3$, and let $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$ denote its tangent, normal and binormal vectors. The parametrization thus satisfies $|\mathbf{\Phi}_\epsilon'(x_1)| = 1$. Let us assume that the function $\mathbf{\Phi}_\epsilon$ is smooth enough and that all the derivatives mentioned below exist. Further, we define the auxiliary functions α_ϵ, β_ϵ, γ_ϵ (corresponding to the usual notions of curvature and torsion) by

$$\alpha_\epsilon := \mathbf{t}_\epsilon' \cdot \mathbf{b}_\epsilon, \; \beta_\epsilon := \mathbf{t}_\epsilon' \cdot \mathbf{n}_\epsilon, \; \gamma_\epsilon := \mathbf{b}_\epsilon' \cdot \mathbf{n}_\epsilon,$$

where $\mathbf{t}_\epsilon'$ is the derivative of $\mathbf{t}_\epsilon$ with respect to x_1, etc. To obtain these relations, we use the assumed orthonormality of the local basis $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$ which gives the orthogonality properties $\mathbf{t}_\epsilon \cdot \mathbf{t}_\epsilon' = 0$, $\mathbf{n}_\epsilon \cdot \mathbf{n}_\epsilon' = 0$, $\mathbf{b}_\epsilon \cdot \mathbf{b}_\epsilon' = 0$, that is $\mathbf{t}_\epsilon'$ may be expressed via $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$ and so on. In this way, we obtain the "laws of motion" of the local frame

$$\begin{aligned} \mathbf{t}_\epsilon' &= \alpha_\epsilon \mathbf{b}_\epsilon + \beta_\epsilon \mathbf{n}_\epsilon, \\ \mathbf{n}_\epsilon' &= -\beta_\epsilon \mathbf{t}_\epsilon - \gamma_\epsilon \mathbf{b}_\epsilon, \\ \mathbf{b}_\epsilon' &= -\alpha_\epsilon \mathbf{t}_\epsilon + \gamma_\epsilon \mathbf{n}_\epsilon. \end{aligned} \tag{4.2.3}$$

At the end, we introduce the mapping $\bar{\mathbf{P}}_\epsilon$

$$\bar{\mathbf{P}}_\epsilon : \Omega_\epsilon \to \mathbb{R}^3, \; \bar{\mathbf{P}}_\epsilon(y) := \mathbf{\Phi}_\epsilon(y_1) + y_2 \mathbf{n}_\epsilon(y_1) + y_3 \mathbf{b}_\epsilon(y_1), \; (y_1, y_2, y_3) \in \Omega_\epsilon, \tag{4.2.4}$$

which gives the parametrization of the curved rod $\widetilde{\Omega}_\epsilon = \bar{\mathbf{P}}_\epsilon(\Omega_\epsilon)$.

To distinguish the derivatives according to domains $\widetilde{\Omega}_\epsilon$, Ω_ϵ and Ω we shall write $\widetilde{\partial}_i = \frac{\partial}{\partial \widetilde{y}_i}$, where $\widetilde{y} = (\widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_3) \in \widetilde{\Omega}_\epsilon$, $\bar{\partial}_i = \frac{\partial}{\partial y_i}$, for $y = (y_1, y_2, y_3) \in \Omega_\epsilon$, and $\partial_i = \frac{\partial}{\partial x_i}$, where $x = (x_1, x_2, x_3) \in \Omega$. In an analogous way, we denote by $\widetilde{v}$ a function defined on $\widetilde{\Omega}_\epsilon$, $\bar{v}$ a function defined on Ω_ϵ and v a function defined on Ω. The respective function spaces used in the thesis are

$$\begin{aligned} V(\widetilde{\Omega}_\epsilon) &:= \{\widetilde{\mathbf{v}} \in W^{1,2}(\widetilde{\Omega}_\epsilon)^3 : \widetilde{\mathbf{v}}|_{\bar{\mathbf{P}}_\epsilon(\{0\} \times \epsilon S)} = \widetilde{\mathbf{v}}|_{\bar{\mathbf{P}}_\epsilon(\{l\} \times \epsilon S)} = 0\}, \\ V(\Omega_\epsilon) &:= \{\bar{\mathbf{v}} \in W^{1,2}(\Omega_\epsilon)^3 : \bar{\mathbf{v}}|_{(\{0\} \times \epsilon S)} = \bar{\mathbf{v}}|_{(\{l\} \times \epsilon S)} = 0\}, \\ V(\Omega) &:= \{\mathbf{v} \in W^{1,2}(\Omega)^3 : \mathbf{v}|_{(\{0\} \times S)} = \mathbf{v}|_{(\{l\} \times S)} = 0\}. \end{aligned}$$

To derive further quantities needed for the transformation of equations from Section 2, we follow the most general approach introduced in [199]. Let us assume for a moment that all derivatives exist. Using the introduced notation we can establish

$$\bar{d}_\epsilon(y) := \det(\bar{\nabla}\bar{\mathbf{P}}_\epsilon(y)) = 1 - \beta_\epsilon(y_1)y_2 - \alpha_\epsilon(y_1)y_3,\ \forall (y_1, y_2, y_3) \in \overline{\Omega}_\epsilon, \tag{4.2.5}$$

where, due to the Jordan property, $\bar{\mathbf{P}}_\epsilon : \Omega_\epsilon \to \widetilde{\Omega}_\epsilon$ is a C^1- diffeomorphism (see Ciarlet [44], Theorem 3.1-1). The covariant basis at point $\bar{\mathbf{P}}_\epsilon(y)$, $y \in \Omega_\epsilon$, of the curved rod is defined by $\bar{\mathbf{g}}_{i,\epsilon}(y) := \bar{\partial}_i \bar{\mathbf{P}}_\epsilon(y)$ and (using (4.2.4)) these vectors are given by

$$\bar{\mathbf{g}}_{1,\epsilon}(y) = (1 - y_2\beta_\epsilon(y_1) - y_3\alpha_\epsilon(y_1))\mathbf{t}_\epsilon(y_1) + y_3\gamma_\epsilon(y_1)\mathbf{n}_\epsilon(y_1) - y_2\gamma_\epsilon(y_1)\mathbf{b}_\epsilon(y_1),$$

$$\bar{\mathbf{g}}_{2,\epsilon}(y) = \mathbf{n}_\epsilon(y_1),\ \bar{\mathbf{g}}_{3,\epsilon}(y) = \mathbf{b}_\epsilon(y_1). \tag{4.2.6}$$

The vectors $\bar{\mathbf{g}}^{j,\epsilon}$ defined by the relations $\bar{\mathbf{g}}_{i,\epsilon} \cdot \bar{\mathbf{g}}^{j,\epsilon} = \delta^{ij}$, constitute the contravariant basis of the curved rod at the point $\bar{\mathbf{P}}_\epsilon(y)$. They have the form

$$\bar{\mathbf{g}}^{1,\epsilon}(y) = \frac{\mathbf{t}_\epsilon(y_1)}{\bar{d}_\epsilon(y)},\ \bar{\mathbf{g}}^{2,\epsilon}(y) = \frac{-y_3\gamma_\epsilon(y_1)\mathbf{t}_\epsilon(y_1)}{\bar{d}_\epsilon(y)} + \mathbf{n}_\epsilon(y_1),$$

$$\bar{\mathbf{g}}^{3,\epsilon}(y) = \frac{y_2\gamma_\epsilon(y_1)\mathbf{t}_\epsilon(y_1)}{\bar{d}_\epsilon(y)} + \mathbf{b}_\epsilon(y_1). \tag{4.2.7}$$

Further, we define the covariant and contravariant metric tensors $(\bar{g}_{ij,\epsilon})_{i,j=1}^3$ and $(\bar{g}^{ij,\epsilon})_{i,j=1}^3$, where

$$\bar{g}_{ij,\epsilon} := \bar{\mathbf{g}}_{i,\epsilon} \cdot \bar{\mathbf{g}}_{j,\epsilon},\ \bar{g}^{ij,\epsilon} := \bar{\mathbf{g}}^{i,\epsilon} \cdot \bar{\mathbf{g}}^{j,\epsilon}. \tag{4.2.8}$$

After substitution $y = \mathbf{R}_\epsilon(x)$, we adopt the notation

$$g^{ij,\epsilon}(x) := \bar{g}^{ij,\epsilon}(\mathbf{R}_\epsilon(x)),\ g_{ij,\epsilon}(x) := \bar{g}_{ij,\epsilon}(\mathbf{R}_\epsilon(x)),\ \mathbf{g}_{i,\epsilon}(x) := \bar{\mathbf{g}}_{i,\epsilon}(\mathbf{R}_\epsilon(x)), \tag{4.2.9}$$

$$\mathbf{g}^{j,\epsilon}(x) := \bar{\mathbf{g}}^{j,\epsilon}(\mathbf{R}_\epsilon(x)),\ d_\epsilon(x) := \bar{d}_\epsilon(\mathbf{R}_\epsilon(x)). \tag{4.2.10}$$

We can also derive the covariant basis at the point $(\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)(x)$, $x \in \Omega$. Thus $\mathbf{o}_{i,\epsilon}(x) := \partial_i(\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)(x)$ and these vectors are given by

$$\mathbf{o}_{1,\epsilon}(x) = (1 - \epsilon x_2\beta_\epsilon(x_1) - \epsilon x_3\alpha_\epsilon(x_1))\mathbf{t}_\epsilon(x_1) + \epsilon x_3\gamma_\epsilon(x_1)\mathbf{n}_\epsilon(x_1) - \epsilon x_2\gamma_\epsilon(x_1)\mathbf{b}_\epsilon(x_1),$$

$$\mathbf{o}_{2,\epsilon}(x) = \epsilon\mathbf{n}_\epsilon(x_1),\ \mathbf{o}_{3,\epsilon}(x) = \epsilon\mathbf{b}_\epsilon(x_1). \tag{4.2.11}$$

The vectors $\mathbf{o}^{j,\epsilon}$ defined by the relations $\mathbf{o}_{i,\epsilon} \cdot \mathbf{o}^{j,\epsilon} = \delta^{ij}$, constitute the contravariant basis at the point $(\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)(x)$, $x \in \Omega$. They have the form

$$\mathbf{o}^{1,\epsilon}(x) = \frac{\mathbf{t}_\epsilon(x_1)}{d_\epsilon(x)},\ \mathbf{o}^{2,\epsilon}(x) = \frac{-x_3\gamma_\epsilon(x_1)\mathbf{t}_\epsilon(x_1)}{d_\epsilon(x)} + \frac{\mathbf{n}_\epsilon(x_1)}{\epsilon},$$

$$\mathbf{o}^{3,\epsilon}(x) = \frac{x_2\gamma_\epsilon(x_1)\mathbf{t}_\epsilon(x_1)}{d_\epsilon(x)} + \frac{\mathbf{b}_\epsilon(x_1)}{\epsilon}. \tag{4.2.12}$$

The respective covariant and contravariant metric tensors can be defined as follows $(o_{ij,\epsilon})_{i,j=1}^3$ and $(o^{ij,\epsilon})_{i,j=1}^3$, where

$$o_{ij,\epsilon} := \mathbf{o}_{i,\epsilon} \cdot \mathbf{o}_{j,\epsilon},\ o^{ij,\epsilon} := \mathbf{o}^{i,\epsilon} \cdot \mathbf{o}^{j,\epsilon}. \tag{4.2.13}$$

These tensors have the form

$$(o_{ij,\epsilon})_{i,j=1}^3 = \begin{pmatrix} d_\epsilon^2 + \epsilon^2 x_3^2\gamma_\epsilon^2 + \epsilon^2 x_2^2\gamma_\epsilon^2 & \epsilon^2 x_3\gamma_\epsilon & -\epsilon^2 x_2\gamma_\epsilon \\ \epsilon^2 x_3\gamma_\epsilon & \epsilon^2 & 0 \\ -\epsilon^2 x_2\gamma_\epsilon & 0 & \epsilon^2 \end{pmatrix} \tag{4.2.14}$$

and

$$(o^{ij,\epsilon})_{i,j=1}^3 = \begin{pmatrix} \frac{1}{d_\epsilon^2} & \frac{-x_3\gamma_\epsilon}{d_\epsilon^2} & \frac{x_2\gamma_\epsilon}{d_\epsilon^2} \\ \frac{-x_3\gamma_\epsilon}{d_\epsilon^2} & \frac{1}{\epsilon^2} + \frac{x_3^2\gamma_\epsilon^2}{d_\epsilon^2} & \frac{-x_2x_3\gamma_\epsilon^2}{d_\epsilon^2} \\ \frac{x_2\gamma_\epsilon}{d_\epsilon^2} & \frac{-x_2x_3\gamma_\epsilon^2}{d_\epsilon^2} & \frac{1}{\epsilon^2} + \frac{x_2^2\gamma_\epsilon^2}{d_\epsilon^2} \end{pmatrix}. \tag{4.2.15}$$

Now, we can calculate

$$o_\epsilon(x) := \sqrt{\det(o_{ij,\epsilon}(x))_{i,j=1}^3} = \epsilon^2 d_\epsilon(x). \tag{4.2.16}$$

For the function $\bar{d}_\epsilon$ introduced in (4.2.5), we can assume that $\bar{d}_\epsilon(y) \neq 0$ for all $y \in \overline{\Omega}_\epsilon$ and "small" ϵ in view of Corollary 4.3.3.

Using the notation above we can also define the function spaces closely related to the curved domains as follows

$$\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l) := \{\langle \mathbf{v}, \psi\rangle \in W_0^{1,2}(0,l)^3 \times\ L^2(0,l) :\ \mathbf{v}' \cdot \mathbf{t} = 0$$
$$\text{and } \mathbf{v}_* := -\psi\mathbf{t} + (\mathbf{v}'\cdot\mathbf{b})\mathbf{n} - (\mathbf{v}'\cdot\mathbf{n})\mathbf{b} \in W_0^{1,2}(0,l)^3\}. \tag{4.2.17}$$

The properties of space $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ will be studied in section 3.4.

4.3 Regularity of the curved rods and their approximations

In Section 3.2 we have established the curved domains. The curved domains depend on ϵ but the dependence need not be only related to the thickness of the domain. We can also assume a more general concept based on a sequence of curved domains that can change their shapes and that converge to a limit curved domain. In the section we thus follow the approach from [199] which enables us to reverse the process, i.e. it enables us to create suitable smooth approximations of a given one-dimensional curved domain. We further show that the one-dimensional domain can exhibit lower regularity than its approximations without a significant influence on the studied problems. In addition, the process also covers the original problem where the shapes remain the same and only the thickness of the domains is changed.

Let us start with several important propositions and their corollaries.

Proposition 4.3.1 [92] *Let us assume we have curve $\mathcal{C}$ given by parametrization $\mathbf{\Phi} \in C^1([0,l])^3$. Then the tangent vector $\mathbf{t} \in C([0,l])^3$ is defined by $\mathbf{t} = \mathbf{\Phi}'$ and there exists a normal vector $\mathbf{n} \in C([0,l])^3$ such that $|\mathbf{n}(x_1)| = 1$, $\mathbf{n}(x_1)\cdot\mathbf{t}(x_1) = 0$, $x_1 \in [0,l]$. The vector $\mathbf{b} = \mathbf{t}\times\mathbf{n}$ has the same regularity properties and completes the local frame.*

Using the result we are able to prove the proposition that enables us to create smooth approximations of a given unit speed curve with low regularity.

Proposition 4.3.2 [199] *Let us assume we have curve $\mathcal{C}$ given by parametrization $\mathbf{\Phi} \in C([0,l])^3$ such that its tangent vector $\mathbf{t} = \mathbf{\Phi}'$ is a piecewise continuous function with a finite set D of points of discontinuity. Then there exist the functions $\mathbf{n}$ and $\mathbf{b}$ piecewise continuous such that*

$$|\mathbf{t}| = |\mathbf{n}| = |\mathbf{b}| = 1,\ \mathbf{t}\perp\mathbf{n}\perp\mathbf{b} \text{ in } [0,l]\setminus D. \tag{4.3.1}$$

In addition, there exist the functions

$$\{\mathbf{\Phi}_\epsilon\}_{\epsilon\in(0,1)},\ \{\mathbf{t}_\epsilon\}_{\epsilon\in(0,1)},\ \{\mathbf{n}_\epsilon\}_{\epsilon\in(0,1)},\ \{\mathbf{b}_\epsilon\}_{\epsilon\in(0,1)} \subset C^\infty([0,l])^3$$

such that $\mathbf{\Phi}_\epsilon$ are parametrizations of Jordan curves $\mathcal{C}_\epsilon$ and

$$\mathbf{\Phi}_\epsilon' = \mathbf{t}_\epsilon,\ |\mathbf{t}_\epsilon| = |\mathbf{n}_\epsilon| = |\mathbf{b}_\epsilon| = 1,\ \mathbf{t}_\epsilon\perp\mathbf{n}_\epsilon\perp\mathbf{b}_\epsilon \text{ on } [0,l] \tag{4.3.2}$$

$$\mathbf{t}_\epsilon \to \mathbf{t},\ \mathbf{n}_\epsilon \to \mathbf{n},\ \mathbf{b}_\epsilon \to \mathbf{b} \textit{ pointwisely in } [0,l] \setminus D, \tag{4.3.3}$$

$$\|\mathbf{t}'_\epsilon\|_\infty,\ \|\mathbf{n}'_\epsilon\|_\infty,\ \|\mathbf{b}'_\epsilon\|_\infty \sim O\left(\frac{1}{\epsilon^r}\right) \tag{4.3.4}$$

and

$$\|\mathbf{t}''_\epsilon\|_\infty,\ \|\mathbf{n}''_\epsilon\|_\infty,\ \|\mathbf{b}''_\epsilon\|_\infty \sim O\left(\frac{1}{\epsilon^{2r}}\right) \tag{4.3.5}$$

for $r \in \left(0, \frac{1}{3}\right)$. *In addition,* $\mathbf{\Phi}_\epsilon \to \mathbf{\Phi}$ *in* $C([0,l])^3$.

In [199] we manage to prove more general result which holds for $\mathbf{\Phi}$ just Lipschitz.

The following corollary gives us an overview of the asymptotic behavior of further quantities defined in the previous section.

Corollary 4.3.3 [199] *Let the functions* $\mathbf{\Phi}_\epsilon$, $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$ *and* $\mathbf{b}_\epsilon$ *have the properties given by Proposition* 4.3.2. *Then the functions* α_ϵ, β_ϵ, γ_ϵ *defined by* (4.2.3) *belong to* $C^\infty([0,l])$ *and have the following behavior*

$$\|\alpha_\epsilon\|_\infty,\ \|\beta_\epsilon\|_\infty,\ \|\gamma_\epsilon\|_\infty \sim O\left(\frac{1}{\epsilon^r}\right), \tag{4.3.6}$$

$$\|\alpha'_\epsilon\|_\infty,\ \|\beta'_\epsilon\|_\infty,\ \|\gamma'_\epsilon\|_\infty \sim O\left(\frac{1}{\epsilon^{2r}}\right), \tag{4.3.7}$$

for $r \in \left(0, \frac{1}{3}\right)$. *In addition,*

$$\sup_{y_1 \in [0,l]} \left(\sup_{(y_2,y_3) \in \epsilon S} |\beta_\epsilon(y_1) y_2 + \alpha_\epsilon(y_1) y_3| \right) < 1 \tag{4.3.8}$$

for ϵ *sufficiently small and thus the mappings* $\bar{\mathbf{P}}_\epsilon$ *defined by* (4.2.4) *are injective and there exist constants* C_j, $j = 0, 1$, *independent of* ϵ *and* x *such that*

$$0 < C_0 \le d_\epsilon(x) \le C_1,\ \forall \epsilon \in (0,1) \textit{ and } \forall x \in \overline{\Omega}. \tag{4.3.9}$$

4.4 Properties of the space $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$

In this section, we summarize and prove the properties of the space $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$.

Proposition 4.4.1 [199] *Let the space* $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ *be defined by* (4.2.17). *Then*

$$\psi = -\mathbf{v}_* \cdot \mathbf{t} \textit{ and } \mathbf{v}(x_1) = \int_0^{x_1} [-(\mathbf{v}_* \cdot \mathbf{b})\mathbf{n} + (\mathbf{v}_* \cdot \mathbf{n})\mathbf{b}]\ dz_1 \tag{4.4.1}$$

for $x_1 \in [0,l]$, *where* ψ *is a piecewise continuous function, and*

$$\mathbf{v}(l) = \int_0^l [-(\mathbf{v}_* \cdot \mathbf{b})\mathbf{n} + (\mathbf{v}_* \cdot \mathbf{n})\mathbf{b}]\ dz_1 = \mathbf{0}. \tag{4.4.2}$$

$\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ *is a nontrivial Hilbert space endowed with the norm*

$$\|\langle \mathbf{v}, \psi \rangle\|^2 := \|\mathbf{v}\|_{1,2}^2 + \|\psi\|_2^2 + \|\mathbf{v}_*\|_{1,2}^2. \tag{4.4.3}$$

Proof: The relations in (4.4.1) follow from (4.2.17) because $\mathbf{v}' \cdot \mathbf{b} = \mathbf{v}_* \cdot \mathbf{n}$ and $-\mathbf{v}' \cdot \mathbf{n} = \mathbf{v}_* \cdot \mathbf{b}$. Relation (4.4.2) is a consequence of the assumed boundary conditions for the function $\mathbf{v}$.

Using the embedding theorem, we obtain from the definition of the functions $\mathbf{v}_*$ and $\mathbf{t}$ (see (4.2.17) and Proposition 4.3.2) and from (4.4.1) that ψ is piecewise continuous.

It is obvious that space $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ is linear and the norm (4.4.3) is induced by the scalar product

$$\langle \mathbf{v},\psi\rangle\cdot\langle \widehat{\mathbf{v}},\widehat{\psi}\rangle := \int_0^l [\mathbf{v}\cdot\widehat{\mathbf{v}}+\mathbf{v}'\cdot\widehat{\mathbf{v}}']\,dx_1+\int_0^l \psi\widehat{\psi}\,dx_1+\int_0^l [\mathbf{v}_*\cdot\widehat{\mathbf{v}}_*+\mathbf{v}_*'\cdot\widehat{\mathbf{v}}_*']\,dx_1 \quad (4.4.4)$$

for arbitrary couples $\langle \mathbf{v},\psi\rangle, \langle \widehat{\mathbf{v}},\widehat{\psi}\rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$.

As a next step, we show that the space $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ is complete in the norm introduced in (4.4.3). Using completeness of the spaces $W_0^{1,2}(0,l)^3$ and $L^2(0,l)$ and taking a Cauchy sequence $\{\langle \mathbf{v}_n,\psi_n\rangle\}_{n=1}^{\infty}$ in $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$, we can find such functions $\mathbf{v}$, $\mathbf{v}_* \in W_0^{1,2}(0,l)^3$ and $\psi \in L^2(0,l)$ that

$$\mathbf{v}_n \to \mathbf{v},\ \ \mathbf{v}_{*,n} \to \mathbf{v}_* \text{ in } W_0^{1,2}(0,l)^3$$

and

$$\psi_n \to \psi \text{ in } L^2(0,l).$$

One can, however, pass to the limit in the norm (4.4.3) and the completeness of $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ is thus proved.

Now, we want to show that the space $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ also contains nontrivial couples. To prove this we take an arbitrary function $\widehat{\mathbf{v}}_* \in W_0^{1,2}(0,l)^3$ such that its components are not identically equal to zero. Then the function $\widehat{\mathbf{v}}$ defined by

$$\widehat{\mathbf{v}}(x_1) := \int_0^{x_1} [-(\widehat{\mathbf{v}}_*\cdot\mathbf{b})\mathbf{n}+(\widehat{\mathbf{v}}_*\cdot\mathbf{n})\mathbf{b}]\,dz_1,\ \ x_1\in[0,l],$$

satisfies

$$\widehat{\mathbf{v}}(l) = \int_0^{l} [-(\widehat{\mathbf{v}}_*\cdot\mathbf{b})\mathbf{n}+(\widehat{\mathbf{v}}_*\cdot\mathbf{n})\mathbf{b}]\,dz_1 = \mathbf{C}_1$$

for some constant vector $\mathbf{C}_1$. Now, we take another function $\mathbf{h}\in W_0^{1,2}(0,l)^3$, which is not proportional with $\widehat{\mathbf{v}}_*$ and whose components are not identically zero, such that

$$\int_0^{l} [-(\mathbf{h}\cdot\mathbf{b})\mathbf{n}+(\mathbf{h}\cdot\mathbf{n})\mathbf{b}]\,dx_1 = \mathbf{C}_2.$$

We define the function $\mathbf{v}_*$ by (we do not use the summation convention here)

$$v_{*,i}(x_1) := \widehat{v}_{*,i}(x_1) - \frac{C_{1,i}}{C_{2,i}}h_i(x_1),\ \ x_1\in[0,l].$$

Then $\mathbf{v}_* \in W_0^{1,2}(0,l)^3$, its components are not identically equal to zero and

$$\mathbf{v}(l) = \int_0^{l} [-(\mathbf{v}_*\cdot\mathbf{b})\mathbf{n}+(\mathbf{v}_*\cdot\mathbf{n})\mathbf{b}]\,dz_1 = \mathbf{0}.$$

This implies that the function $\mathbf{v}$ defined by

$$\mathbf{v}(x_1) = \int_0^{x_1} [-(\mathbf{v}_*\cdot\mathbf{b})\mathbf{n}+(\mathbf{v}_*\cdot\mathbf{n})\mathbf{b}]\,dz_1,\ \ x_1\in[0,l],$$

belongs to $W_0^{1,2}(0,l)^3$, $\mathbf{v}'\cdot\mathbf{t}=0$, $\psi=-\mathbf{v}_*\cdot\mathbf{t}$ is piecewise continuous and thus the nontrivial couple $\langle \mathbf{v},\psi\rangle$ belongs to $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$. □

Now, we construct spaces that approximate the space $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ via tangent, normal, and binormal vectors.

Proposition 4.4.2 [199] *Let* $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$ *and* $\mathbf{b}_\epsilon$ *be the functions from Proposition* 4.3.2 *and let the spaces* $\mathcal{V}_0^{\mathbf{t}_\epsilon,\mathbf{n}_\epsilon,\mathbf{b}_\epsilon}(0,l)$ *be defined by* (4.2.17) *using the functions* $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$ *instead of* $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$. *Let, further,* $\langle \mathbf{v}, \psi\rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$. *Then there exist couples* $\langle \mathbf{v}_\epsilon, \psi_\epsilon\rangle \in \mathcal{V}_0^{\mathbf{t}_\epsilon,\mathbf{n}_\epsilon,\mathbf{b}_\epsilon}(0,l)$ *generating the functions* $\mathbf{v}_{*,\epsilon}$ *such that*

$$\{\mathbf{v}_\epsilon\}_{\epsilon\in(0,1)} \subset C_0^\infty(0,l)^3,\ \{\psi_\epsilon\}_{\epsilon\in(0,1)} \subset C_0^\infty(0,l),\ \{\mathbf{v}_{*,\epsilon}\}_{\epsilon\in(0,1)} \subset C_0^\infty(0,l)^3,$$

$$\mathbf{v}_\epsilon \to \mathbf{v},\ \mathbf{v}_{*,\epsilon} \to \mathbf{v}_* \ \textit{in}\ W_0^{1,2}(0,l)^3, \tag{4.4.5}$$

$$\psi_\epsilon \to \psi \ \textit{pointwisely in}\ [0,l]\setminus D \ \textit{and in}\ L^p(0,l),\ \forall p\in[1,\infty), \tag{4.4.6}$$

for $\epsilon \to 0$, *and*

$$\|\mathbf{v}_\epsilon''\|_2 \sim O\left(\frac{1}{\epsilon^r}\right),\ \|\psi_\epsilon'\|_2 \sim O\left(\frac{1}{\epsilon^r}\right),\ r\in\left(0,\frac{1}{3}\right). \tag{4.4.7}$$

Proof: In the definition (4.2.17) of the space $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$, we defined the function $\mathbf{v}_*$ with the help of the function $\mathbf{v}$. But we can use the inverse procedure as in the proof of Proposition 4.4.1. We can then easily construct by regularization the set of functions $\{\widehat{\mathbf{v}}_{*,\epsilon}\}_{\epsilon\in(0,1)} \subset C_0^\infty(0,l)^3$ such that

$$\widehat{\mathbf{v}}_{*,\epsilon} \to \mathbf{v}_* \text{ in } W_0^{1,2}(0,l)^3 \text{ for } \epsilon\to 0.$$

We know from Proposition 4.3.2 that $\mathbf{t}_\epsilon \to \mathbf{t}$, $\mathbf{n}_\epsilon \to \mathbf{n}$ and $\mathbf{b}_\epsilon \to \mathbf{b}$ pointwisely in $[0,l]\setminus D$ and strongly in $L^p(0,l)^3$, $p\in[1,\infty)$, and thus, using the Lebesgue theorem,

$$\int_0^l [-(\widehat{\mathbf{v}}_{*,\epsilon}\cdot\mathbf{b}_\epsilon)\mathbf{n}_\epsilon + (\widehat{\mathbf{v}}_{*,\epsilon}\cdot\mathbf{n}_\epsilon)\mathbf{b}_\epsilon]\, dz_1 = \mathbf{C}_3(\epsilon) \to \mathbf{0},$$

for $\epsilon\to 0$. Let $\mathbf{h}$ be some vector function from $C_0^\infty(0,l)^3$ that is not proportional to $\widehat{\mathbf{v}}_*$ and whose components are not identically equal to zero, such that

$$\int_0^l [-(\mathbf{h}\cdot\mathbf{b})\mathbf{n} + (\mathbf{h}\cdot\mathbf{n})\mathbf{b}]\, dz_1 = \mathbf{C}_4 = (C_{4,1}, C_{4,2}, C_{4,3}),$$

with $C_{4,i}\neq 0$, $i=1,2,3$. Then

$$\int_0^l [-(\mathbf{h}\cdot\mathbf{b}_\epsilon)\mathbf{n}_\epsilon + (\mathbf{h}\cdot\mathbf{n}_\epsilon)\mathbf{b}_\epsilon]\, dz_1 = \mathbf{C}_4 + \mathbf{C}_5(\epsilon),$$

where $\mathbf{C}_5(\epsilon)\to\mathbf{0}$ for $\epsilon\to 0$. Now, we define functions $\mathbf{v}_{*,\epsilon}$ by (we do not use the summation convention here)

$$v_{*,\epsilon,i}(x_1) := \widehat{v}_{*,\epsilon,i}(x_1) - \frac{C_{3,i}(\epsilon)}{C_{4,i}+C_{5,i}(\epsilon)} h_i(x_1),\ x_1\in[0,l],\ i=1,2,3. \tag{4.4.8}$$

Then $\mathbf{v}_{*,\epsilon}\in C_0^\infty(0,l)^3$, the functions $v_{*,\epsilon,i}$, $i=1,2,3$, are not identically zero and

$$\int_0^l [-(\mathbf{v}_{*,\epsilon}\cdot\mathbf{b}_\epsilon)\mathbf{n}_\epsilon + (\mathbf{v}_{*,\epsilon}\cdot\mathbf{n}_\epsilon)\mathbf{b}_\epsilon]\, dz_1 = \mathbf{0}.$$

Then analogously as in Proposition 4.4.1 we define the functions

$$\mathbf{v}_\epsilon(x_1) := \int_0^{x_1} [-(\mathbf{v}_{*,\epsilon}\cdot\mathbf{b}_\epsilon)\mathbf{n}_\epsilon + (\mathbf{v}_{*,\epsilon}\cdot\mathbf{n}_\epsilon)\mathbf{b}_\epsilon]\, dz_1, \tag{4.4.9}$$

$$\psi_\epsilon := -\mathbf{v}_{*,\epsilon}\cdot\mathbf{t}_\epsilon \tag{4.4.10}$$

and thus $\langle \mathbf{v}_\epsilon, \psi_\epsilon \rangle \in \mathcal{V}_0^{\mathbf{t}_\epsilon,\mathbf{n}_\epsilon,\mathbf{b}_\epsilon}(0,l)$. Since $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$ and $\mathbf{b}_\epsilon \in C^\infty([0,l])^3$, we get easily from (4.4.8) and from the properties of the functions $\widehat{\mathbf{v}}_{*,\epsilon}$ and $\mathbf{h}$ that $\mathbf{v}_{*,\epsilon} \in C_0^\infty(0,l)^3$ and thus using (4.4.9), (4.4.10), $\mathbf{v}_\epsilon \in C_0^\infty(0,l)^3$ and $\psi_\epsilon \in C_0^\infty(0,l)$ for all $\epsilon \in (0,1)$.

The verification of (4.4.5) and (4.4.6) can be done easily and we omit it. From (4.4.9), it follows the estimate

$$
\begin{aligned}
\|\mathbf{v}_\epsilon''\|_2 &= \| - (\mathbf{v}_{*,\epsilon}' \cdot \mathbf{b}_\epsilon)\mathbf{n}_\epsilon - (\mathbf{v}_{*,\epsilon} \cdot \mathbf{b}_\epsilon')\mathbf{n}_\epsilon - (\mathbf{v}_{*,\epsilon} \cdot \mathbf{b}_\epsilon)\mathbf{n}_\epsilon' + \\
&+ (\mathbf{v}_{*,\epsilon}' \cdot \mathbf{n}_\epsilon)\mathbf{b}_\epsilon + (\mathbf{v}_{*,\epsilon} \cdot \mathbf{n}_\epsilon')\mathbf{b}_\epsilon + (\mathbf{v}_{*,\epsilon} \cdot \mathbf{n}_\epsilon)\mathbf{b}_\epsilon'\|_2 \le \\
&\le C\Big(\|\mathbf{v}_{*,\epsilon}'\|_2 + \|\mathbf{v}_{*,\epsilon}\|_2(\|\mathbf{b}_\epsilon'\|_\infty + \|\mathbf{n}_\epsilon'\|_\infty)\Big),
\end{aligned}
$$

which, together with (4.3.4), yields the first relation in (4.4.7). The second relation in (4.4.7) easily follows from the fact that $\psi_\epsilon = -\mathbf{v}_{*,\epsilon} \cdot \mathbf{t}_\epsilon$ and from (4.3.4). □

4.5 Shells

In this subsection, we introduce the notation used in Section 5. The notation is related to shells and is used in the limit processes which leads to two-dimensional models.

The domain $\widetilde{\Omega}_\epsilon \subset \mathbb{R}^3$ is defined by the use of a reference domain $\Omega = S \times (0,1)$, $S \subset \mathbb{R}^2$, $\partial S \in C^{0,1}$, and the mapping $\Theta_\epsilon : \Omega \to \widetilde{\Omega}_\epsilon$ so that

$$
\Theta_\epsilon : (x_1, x_2, x_3) \to \boldsymbol{\theta}(x_1, x_2) + \epsilon x_3 \mathbf{a}_3(x_1, x_2), \tag{4.5.1}
$$

where $\boldsymbol{\theta} : S \to \mathbb{R}^3$ and

$$
\begin{aligned}
\mathbf{a}_1 &:= (\partial_1\theta_1, \partial_1\theta_2, \partial_1\theta_3), \\
\mathbf{a}_2 &:= (\partial_2\theta_1, \partial_2\theta_2, \partial_2\theta_3), \\
\mathbf{a}_3 &:= \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|}.
\end{aligned}
$$

We suppose that $\mathbf{a}_j$, $\partial_\alpha \mathbf{a}_j$ and $\partial^2_{\alpha\beta}\mathbf{a}_3 \in L^\infty(\Omega)^3$, where α, $\beta = 1,2$ and $j = 1,2,3$.

First, we define the covariant basis (see [44], section 1.2)

$$
\begin{aligned}
\mathbf{g}_{1,\epsilon} &:= \partial_1\Theta_\epsilon = \mathbf{a}_1 + \epsilon x_3 \partial_1 \mathbf{a}_3, && (4.5.2)\\
\mathbf{g}_{2,\epsilon} &:= \partial_2\Theta_\epsilon = \mathbf{a}_2 + \epsilon x_3 \partial_2 \mathbf{a}_3, && (4.5.3)\\
\mathbf{g}_{3,\epsilon} &:= \partial_3\Theta_\epsilon = \epsilon \mathbf{a}_3, && (4.5.4)
\end{aligned}
$$

the covariant metric tensor G_ϵ

$$
[G_\epsilon]_{ij} = g_{ij,\epsilon} := \mathbf{g}_{i,\epsilon} \cdot \mathbf{g}_{j,\epsilon}, \tag{4.5.5}
$$

and its determinant $g_\epsilon := \det(\mathrm{G}_\epsilon)$. Further, we also define the contravariant basis by the relations

$$
\mathbf{g}^{i,\epsilon} \cdot \mathbf{g}_{j,\epsilon} = \delta^{ij}. \tag{4.5.6}
$$

It is known from [44], Theorem 1.2-1, that

$$
\left[G_\epsilon^{-1}\right]^{ij} = g^{ij,\epsilon} := \mathbf{g}^{i,\epsilon} \cdot \mathbf{g}^{j,\epsilon}
$$

and also (see [44], proof of Theorem 1.3) that

$$
[\mathbf{g}^{i,\epsilon}(x)]_k = \widetilde{\partial}_k \Theta_{i,\epsilon}^{-1}(\widetilde{x}),
$$

where $\widetilde{x} = \Theta_\epsilon(x)$.

For convenience, we denote the determinant of submatrix $(G_\epsilon)_{i,j=1}^2$ as d_ϵ. Relations (4.5.2)–(4.5.4) and (4.5.6) enable us to express the contravariant basis:

$$\begin{aligned}
\mathbf{g}^{1,\epsilon} &= d_\epsilon^{-1}\left(|\mathbf{g}_{2,\epsilon}|^2\mathbf{g}_{1,\epsilon} - (\mathbf{g}_{1,\epsilon}\cdot\mathbf{g}_{2,\epsilon})\mathbf{g}_{2,\epsilon}\right),\\
\mathbf{g}^{2,\epsilon} &= d_\epsilon^{-1}\left(|\mathbf{g}_{1,\epsilon}|^2\mathbf{g}_{2,\epsilon} - (\mathbf{g}_{1,\epsilon}\cdot\mathbf{g}_{2,\epsilon})\mathbf{g}_{1,\epsilon}\right),\\
\mathbf{g}^{3,\epsilon} &= \epsilon^{-1}\mathbf{a}_3.
\end{aligned} \tag{4.5.7}$$

The contravariant basis is well-defined, because $d_\epsilon > 0$ (see (4.5.14)). For further calculations, we determine explicitly also the matrix G_ϵ and its inverse:

$$G_\epsilon := \begin{pmatrix} g_{11,\epsilon} & g_{12,\epsilon} & 0\\ . & g_{22,\epsilon} & 0\\ \mathrm{sym} & . & \epsilon^2 \end{pmatrix}, \quad G_\epsilon^{-1} = \begin{pmatrix} g^{11,\epsilon} & g^{12,\epsilon} & 0\\ . & g^{22,\epsilon} & 0\\ \mathrm{sym} & . & \epsilon^{-2} \end{pmatrix},$$

where

$$\begin{aligned}
g_{11,\epsilon} &= |\mathbf{a}_1|^2 + 2\epsilon x_3\mathbf{a}_1\cdot\partial_1\mathbf{a}_3 + \epsilon^2 x_3^2|\partial_1\mathbf{a}_3|^2,\\
g_{12,\epsilon} &= \mathbf{a}_1\cdot\mathbf{a}_2 + \epsilon x_3\left(\mathbf{a}_1\cdot\partial_2\mathbf{a}_3 + \mathbf{a}_2\cdot\partial_1\mathbf{a}_3\right) + \epsilon^2 x_3^2\partial_1\mathbf{a}_3\cdot\partial_2\mathbf{a}_3,\\
g_{22,\epsilon} &= |\mathbf{a}_2|^2 + 2\epsilon x_3\mathbf{a}_2\cdot\partial_2\mathbf{a}_3 + \epsilon^2 x_3^2|\partial_2\mathbf{a}_3|^2,\\
g^{11,\epsilon} &= g_{22,\epsilon}d_\epsilon^{-1},\\
g^{12,\epsilon} &= -g_{12,\epsilon}d_\epsilon^{-1},\\
g^{22,\epsilon} &= g_{11,\epsilon}d_\epsilon^{-1}.
\end{aligned}$$

Terms $g_{13,\epsilon}$ and $g_{23,\epsilon}$ are equal to zero because

$$\mathbf{g}_{1,\epsilon}\cdot\mathbf{g}_{3,\epsilon} = \epsilon\mathbf{a}_1\cdot\mathbf{a}_3 + \epsilon^2 x_3\mathbf{a}_3\cdot\partial_2\mathbf{a}_3 = 0.$$

The last equality is due to orthogonality of $\mathbf{a}_1$ and $\mathbf{a}_3$, and equality $\mathbf{a}_3\cdot\partial_2\mathbf{a}_3 = \frac{1}{2}\partial_2|\mathbf{a}_3|^2 = \frac{1}{2}\partial_2 1 = 0$. Similarly, $g_{23,\epsilon} = 0$ and thus also $g^{13,\epsilon} = g^{23,\epsilon} = 0$.

Mapping Θ_ϵ can be decomposed into two parts: deformation and contraction. Therefore, matrix G_ϵ, as well as the inverse matrix G_ϵ^{-1}, can be decomposed into two parts. In Section 5, we need the decomposition of G_ϵ^{-1}. Thus, we denote

$$E_\epsilon := \begin{pmatrix} 1 & 0 & 0\\ . & 1 & 0\\ \mathrm{sym} & . & \epsilon^{-1} \end{pmatrix}, \tag{4.5.8}$$

$$R_\epsilon := \left([\mathbf{g}^{1,\epsilon}]^T, [\mathbf{g}^{2,\epsilon}]^T, [\mathbf{a}_3]^T\right) = \begin{pmatrix} [\mathbf{g}^{1,\epsilon}]_1 & [\mathbf{g}^{2,\epsilon}]_1 & [\mathbf{a}_3]_1\\ [\mathbf{g}^{1,\epsilon}]_2 & [\mathbf{g}^{2,\epsilon}]_2 & [\mathbf{a}_3]_2\\ [\mathbf{g}^{1,\epsilon}]_3 & [\mathbf{g}^{2,\epsilon}]_3 & [\mathbf{a}_3]_3 \end{pmatrix}. \tag{4.5.9}$$

It holds that $G_\epsilon^{-1} = E_\epsilon R_\epsilon^T R_\epsilon E_\epsilon$. It is an easy matter to demonstrate $\det(R_\epsilon^T R_\epsilon) = d_\epsilon^{-1}$, $g_\epsilon = d_\epsilon\epsilon^2$ and

$$R_\epsilon^T R_\epsilon = \begin{pmatrix} g^{11,\epsilon} & g^{12,\epsilon} & 0\\ . & g^{22,\epsilon} & 0\\ \mathrm{sym} & . & 1 \end{pmatrix}. \tag{4.5.10}$$

From the relations (4.5.2)–(4.5.7), it is simple to prove that $R_\epsilon^T R_\epsilon$ is a symmetric positive definite matrix. Hence, $d_\epsilon^{-1} > 0$ and therefore $\sqrt{d_\epsilon} > 0$ is well-defined. Furthermore, it stems from Cauchy's inequality that d_ϵ would equal zero if and only if $\mathbf{g}_{1,\epsilon} = \mathbf{g}_{2,\epsilon}$. However, this situation cannot occur because $\mathbf{g}_{1,\epsilon}$ and $\mathbf{g}_{2,\epsilon}$ are linearly independent.

In the end, we have a look at the asymptotic behavior of the above-defined scalars and vectors for $\epsilon \to 0$. We have

$$\mathbf{g}_{1,\epsilon} \to \mathbf{a}_1, \quad \mathbf{g}_{2,\epsilon} \to \mathbf{a}_2, \quad \mathbf{g}_{3,\epsilon} \to \mathbf{0} \text{ in } W^{1,\infty}(\Omega)^3, \tag{4.5.11}$$

$$G_\epsilon \to G = \begin{pmatrix} |\mathbf{a}_1|^2 & \mathbf{a}_1 \cdot \mathbf{a}_2 & 0 \\ . & |\mathbf{a}_2|^2 & 0 \\ \text{sym} & . & 0 \end{pmatrix} \text{ in } L^\infty(\Omega)^9, \tag{4.5.12}$$

$$R_\epsilon^T R_\epsilon \to R^T R = \begin{pmatrix} d^{-1}|\mathbf{a}_2|^2 & -d^{-1}\mathbf{a}_1 \cdot \mathbf{a}_2 & 0 \\ . & d^{-1}|\mathbf{a}_1|^2 & 0 \\ \text{sym} & . & 1 \end{pmatrix} \quad \text{in } L^\infty(\Omega)^9, \tag{4.5.13}$$

$$g_\epsilon \to 0, \quad d_\epsilon \to d = |\mathbf{a}_1|^2|\mathbf{a}_2|^2 - (\mathbf{a}_1 \cdot \mathbf{a}_2)^2 \text{ in } L^\infty(\Omega). \tag{4.5.14}$$

Therefore, $d_\epsilon \geq \delta > 0$ and the contravariant basis is well-defined by relations (4.5.7). It immediately follows from the above convergences that

$$\mathbf{g}^{1,\epsilon} \quad \to \quad \mathbf{a}^1 := d^{-1}\left(|\mathbf{a}_2|^2\mathbf{a}_1 - (\mathbf{a}_1 \cdot \mathbf{a}_2)\mathbf{a}_2\right) \text{ in } W^{1,\infty}(\Omega)^3, \tag{4.5.15}$$

$$\mathbf{g}^{2,\epsilon} \quad \to \quad \mathbf{a}^2 := d^{-1}\left(|\mathbf{a}_1|^2\mathbf{a}_2 - (\mathbf{a}_1 \cdot \mathbf{a}_2)\mathbf{a}_1\right) \text{ in } W^{1,\infty}(\Omega)^3. \tag{4.5.16}$$

Hence,

$$R = \begin{pmatrix} [\mathbf{a}^1]_1 & [\mathbf{a}^2]_1 & [\mathbf{a}_3]_1 \\ [\mathbf{a}^1]_2 & [\mathbf{a}^2]_2 & [\mathbf{a}_3]_2 \\ [\mathbf{a}^1]_3 & [\mathbf{a}^2]_3 & [\mathbf{a}_3]_3 \end{pmatrix}. \tag{4.5.17}$$

Let us remark that all these limit functions depend only on x_1 and x_2 because $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ as well as $\mathbf{a}^1$, $\mathbf{a}^2$ are independent of x_3.

5 Elasticity

In this section, we study two problems related to elastic materials. In the first subsection, we show how to treat dimension reduction in the case of a time-dependent model of linear elasticity. The next subsection deals with a more general model related to heat-elastic materials. The results were published in [210] and [215]. Both results are related to curved rods. The technique, that enables us to derive limit models, is based upon a priori estimates and respective convergences applied to higher dimensional models.

Let us, however, start with a brief overview of related results. The most straightforward results are for cylinders [1] because in this case, we do not need to work with curved domains. The situation starts to be complicated if some kind of deformation comes into play. The deformation is related to the functions $\mathbf{\Phi}_\epsilon$ (see Section 3.2), where ϵ corresponds to the thickness of the domain. It is obvious that the regularity of the function is important for various approaches and one of the main aims is to relax the regularity assumptions. The most natural approach is to assume that $\mathbf{\Phi}_\epsilon$ is independent of ϵ. In this case, C^4-regularity of $\mathbf{\Phi}$ was assumed in [99]. The regularity was further relaxed to C^3 in [7], [100] and [195]. The next question is whether also the shapes of the curved rods can depend on ϵ. If it is possible we can think about the relaxation of the regularity of a limit function $\mathbf{\Phi}$. The problem was studied in [199], where, using the more general framework, we start with C^1 piecewise continuous parametrization $\mathbf{\Phi}$ which corresponds to the limit curved rod. The function $\mathbf{\Phi}$ can be approximated with smoother parametrizations $\mathbf{\Phi}_\epsilon$ representing Jordan curves that are used for three-dimensional models. The main idea of the approximation is based on a similar approach for shells from [28]. It was obviously necessary to modify the approach for curved rods. The approximation will be used in all subsequent sections.

5.1 A general asymptotic dynamic model for elastic curved rods

In this subsection, we introduce a linear evolution problem for clamped curved rods and show the limit process for $\epsilon \to 0$ under minimal regularity assumptions on the geometry. The whole process of dimension reduction is decomposed into several subsections covering a weak formulation of the problem and its transformation on a referential domain, basic assumptions, auxiliary propositions, a priori estimates, the limit process, and qualitative properties of limit functions.

5.1.1 Weak formulation of an evolution equation for the curved rods and the main result

In this subsection, we introduce the weak formulation of the linear evolution model for the elastic curved rods. Let us assume that we have a sequence of elastic curved rods represented by domains $\widetilde{\Omega}_\epsilon$. $\widetilde{\Omega}_\epsilon$ are defined by mappings $\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon$ (see (4.2.2) and (4.2.4)) for $\epsilon \in (0,1)$ arbitrary but fixed as three-dimensional homogeneous and isotropic elastic bodies with the Lamé constants $\lambda \geq 0$, $\mu > 0$ and with mass density $\widetilde{\rho}_\epsilon$. Assume further that $\widetilde{\Omega}_\epsilon$ is clamped on both bases $\bar{\mathbf{P}}_\epsilon(\{0\} \times \epsilon S)$ and $\bar{\mathbf{P}}_\epsilon(\{l\} \times \epsilon S)$. Let $\widetilde{\mathbf{f}}_\epsilon$ be the body force and $\widetilde{\mathbf{h}}_\epsilon$ the surface traction acting on the curved rods $\widetilde{\Omega}_\epsilon$ such that $\widetilde{\mathbf{f}}_\epsilon \in L^2(0,T;L^2(\widetilde{\Omega}_\epsilon)^3)$ and $\widetilde{\mathbf{h}}_\epsilon \in W^{1,1}(0,T;L^2((\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)((0,l) \times \partial S))^3)$, for $\epsilon \in (0,1)$. The equilibrium displacement $\widetilde{\mathbf{u}}_\epsilon$ is a weak solution of the equation

$$\int_0^T \int_{\widetilde{\Omega}_\epsilon} [-\widetilde{\rho}_\epsilon \partial_t \widetilde{\mathbf{u}}_\epsilon(t) \cdot \partial_t \widetilde{\mathbf{v}}(t) + \widetilde{A}^{ijkl} D_{kl} \widetilde{\mathbf{u}}_\epsilon(t) D_{ij} \widetilde{\mathbf{v}}(t)]\, d\widetilde{y} dt =$$

$$= \int_0^T \int_{\widetilde{\Omega}_\epsilon} \widetilde{\mathbf{f}}_\epsilon(t) \cdot \widetilde{\mathbf{v}}(t)\, d\widetilde{y} dt + \int_0^T \int_{\widetilde{S}_\epsilon} \widetilde{\mathbf{h}}_\epsilon(t) \cdot \widetilde{\mathbf{v}}(t)\, d\widetilde{S}_\epsilon dt \qquad (5.1.1)$$

for all $\widetilde{\mathbf{v}} \in C_0^\infty(0,T;V(\widetilde{\Omega}_\epsilon))$, where $\widetilde{S}_\epsilon := (\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)((0,l) \times \partial S)$, $\widetilde{A}^{ijkl} := \lambda \delta^{ij} \delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})$ and $D\widetilde{\mathbf{v}} = (D_{ij}\widetilde{\mathbf{v}})_{i,j=1}^3$ stands for the symmetric part of the gradient of the function $\widetilde{\mathbf{v}}$. The solution $\widetilde{\mathbf{u}}_\epsilon$ satisfies the initial state

$$\widetilde{\mathbf{u}}_\epsilon(\widetilde{y},0) = \widetilde{\mathbf{q}}_{0,\epsilon}(\widetilde{y}),\ \partial_t \widetilde{\mathbf{u}}_\epsilon(\widetilde{y},0) = \widetilde{\mathbf{q}}_{1,\epsilon}(\widetilde{y}),\ \widetilde{y} \in \widetilde{\Omega}_\epsilon. \qquad (5.1.2)$$

Based on Section 3.3, mappings $\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon$ are parametrizations of smooth three-dimensional curved rods with corresponding vectors $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon \in C^\infty([0,l])^3$.

We can transform equation (5.1.1) on a referential domain Ω. Using the notation $\mathbf{u}_\epsilon := \widetilde{\mathbf{u}}_\epsilon(\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)$, $\rho_\epsilon := \widetilde{\rho}_\epsilon(\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)$, $\mathbf{v}_\epsilon := \widetilde{\mathbf{v}}(\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)$, $\mathbf{q}_{0,\epsilon} := \widetilde{\mathbf{q}}_{0,\epsilon}(\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)$, $\mathbf{q}_{1,\epsilon} := \widetilde{\mathbf{q}}_{1,\epsilon}(\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)$, $\mathbf{f}_\epsilon := \widetilde{\mathbf{f}}_\epsilon(\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)$ and $\mathbf{h}_\epsilon := \widetilde{\mathbf{h}}_\epsilon(\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)$, we can rewrite (5.1.1)–(5.1.2) as follows

$$\int_0^T \int_\Omega [-\rho_\epsilon \partial_t \mathbf{u}_\epsilon(t) \cdot \partial_t \mathbf{v}(t) + A_\epsilon^{ijkl} \omega_{kl}^\epsilon(\mathbf{u}_\epsilon(t)) \omega_{ij}^\epsilon(\mathbf{v}(t))]\, d_\epsilon\, dxdt =$$

$$= \int_0^T \int_\Omega \mathbf{f}_\epsilon(t) \cdot \mathbf{v}(t)\, d_\epsilon\, dxdt + \int_0^T \int_0^l \int_{\partial S} \mathbf{h}_\epsilon(t) \cdot \mathbf{v}(t)\, d_\epsilon \sqrt{\nu_i o^{ij,\epsilon} \nu_j}\, dS dx_1 dt \quad (5.1.3)$$

for all $\mathbf{v} \in C_0^\infty(0,T;V(\Omega))$, where the solution $\mathbf{u}_\epsilon$ satisfies the initial state

$$\mathbf{u}_\epsilon(x,0) = \mathbf{q}_{0,\epsilon}(x),\ \partial_t \mathbf{u}_\epsilon(x,0) = \mathbf{q}_{1,\epsilon}(x), \qquad (5.1.4)$$

ν_i, $i = 1,2,3$, are the components of the unit outward normal to $(0,l) \times \partial S$, $(o^{ij,\epsilon})_{i,j=1}^3$ was introduced in (4.2.15). Symmetric tensor $\omega^\epsilon(\mathbf{v})$ has the form

$$\omega^\epsilon(\mathbf{v}) = \frac{1}{\epsilon}\theta^\epsilon(\mathbf{v}) + \kappa^\epsilon(\mathbf{v}). \qquad (5.1.5)$$

The symmetric tensors θ^ϵ and κ^ϵ consist of several nonzero elements that are summarized below.

$$\theta^\epsilon_{12}(\mathbf{v}) = \frac{1}{2}(\partial_2\mathbf{v}\cdot\mathbf{g}_{1,\epsilon}),\ \theta^\epsilon_{22}(\mathbf{v}) = \partial_2\mathbf{v}\cdot\mathbf{n}_\epsilon,\ \theta^\epsilon_{33}(\mathbf{v}) = \partial_3\mathbf{v}\cdot\mathbf{b}_\epsilon, \tag{5.1.6}$$

$$\theta^\epsilon_{13}(\mathbf{v}) = \frac{1}{2}(\partial_3\mathbf{v}\cdot\mathbf{g}_{1,\epsilon}),\ \theta^\epsilon_{23}(\mathbf{v}) = \frac{1}{2}(\partial_2\mathbf{v}\cdot\mathbf{b}_\epsilon + \partial_3\mathbf{v}\cdot\mathbf{n}_\epsilon), \tag{5.1.7}$$

$$\kappa^\epsilon_{11}(\mathbf{v}) = \partial_1\mathbf{v}\cdot\mathbf{g}_{1,\epsilon},\ \kappa^\epsilon_{12}(\mathbf{v}) = \frac{1}{2}(\partial_1\mathbf{v}\cdot\mathbf{n}_\epsilon),\ \kappa^\epsilon_{13}(\mathbf{v}) = \frac{1}{2}(\partial_1\mathbf{v}\cdot\mathbf{b}_\epsilon). \tag{5.1.8}$$

Since the transformation of the symmetric parts of the gradients from (5.1.1) is a crucial step in derivation of (5.1.3), we introduce here its main ideas. For more details, we refer the reader to [44] and [199]. Let us take function $\widetilde{\mathbf{v}} \in W^{1,2}(\widetilde{\Omega}_\epsilon)^3$. First, we show how the transformation between $\widetilde{\Omega}_\epsilon$ and Ω_ϵ looks like. We take $\bar{\mathbf{P}}_\epsilon$ defined in (4.2.4), $\bar{\mathbf{g}}^{k,\epsilon} = ([\bar{\mathbf{g}}^{k,\epsilon}]_1, [\bar{\mathbf{g}}^{k,\epsilon}]_2, [\bar{\mathbf{g}}^{k,\epsilon}]_3)$ from (4.2.7) and $\bar{\mathbf{w}}_\epsilon = (\bar{w}_{1,\epsilon}, \bar{w}_{2,\epsilon}, \bar{w}_{3,\epsilon})$ are such that

$$\widetilde{v}_i \circ \bar{\mathbf{P}}_\epsilon = \bar{w}_{k,\epsilon}[\bar{\mathbf{g}}^{k,\epsilon}]_i,\ i = 1,2,3.$$

Then

$$(\widetilde{\partial_j}\widetilde{v}_i) \circ \bar{\mathbf{P}}_\epsilon = (\bar{\partial}_l \bar{w}_{k,\epsilon} - \bar{w}_{q,\epsilon}\Gamma^q_{lk,\epsilon})[\bar{\mathbf{g}}^{k,\epsilon}]_i[\bar{\mathbf{g}}^{l,\epsilon}]_j,$$

where the Christoffel symbols $\Gamma^i_{jk,\epsilon}$ are defined by

$$\Gamma^i_{jk,\epsilon} = \bar{\mathbf{g}}^{i,\epsilon}\cdot\bar{\partial}_j\bar{\mathbf{g}}_{k,\epsilon},\ i,j,k = 1,2,3.$$

Using the notation

$$D_{i||j}\bar{\mathbf{w}}_\epsilon := \frac{1}{2}(\bar{\partial}_i\bar{w}_{j,\epsilon} + \bar{\partial}_j\bar{w}_{i,\epsilon}) - \bar{w}_{p,\epsilon}\Gamma^p_{ij,\epsilon},$$

we obtain

$$D_{ij}(\widetilde{\mathbf{v}}) \circ \bar{\mathbf{P}}_\epsilon = D_{k||l}\bar{\mathbf{w}}_\epsilon[\bar{\mathbf{g}}^{k,\epsilon}]_i[\bar{\mathbf{g}}^{l,\epsilon}]_j,\ i,j = 1,2,3.$$

Now, we define the vector function $\bar{\mathbf{v}}_\epsilon$ by

$$\bar{\mathbf{v}}_\epsilon := \bar{w}_{i,\epsilon}\bar{\mathbf{g}}^{i,\epsilon}(= \widetilde{\mathbf{v}} \circ \bar{\mathbf{P}}_\epsilon).$$

Then

$$\begin{aligned}
\frac{1}{2}(\bar{\partial}_i\bar{w}_{j,\epsilon} + \bar{\partial}_j\bar{w}_{i,\epsilon}) &= \frac{1}{2}\Big(\bar{\partial}_i(\bar{w}_{k,\epsilon}\bar{\mathbf{g}}^{k,\epsilon}\cdot\bar{\mathbf{g}}_{j,\epsilon}) + \bar{\partial}_j(\bar{w}_{l,\epsilon}\bar{\mathbf{g}}^{l,\epsilon}\cdot\bar{\mathbf{g}}_{i,\epsilon})\Big) = \\
&= \frac{1}{2}\Big(\bar{\partial}_i(\bar{\mathbf{v}}_\epsilon\cdot\bar{\mathbf{g}}_{j,\epsilon}) + \bar{\partial}_j(\bar{\mathbf{v}}_\epsilon\cdot\bar{\mathbf{g}}_{i,\epsilon})\Big).
\end{aligned}$$

Since the Christoffel symbols are symmetric in the indices i, j, we get, using the identities

$$\begin{aligned}
\bar{w}_{k,\epsilon}\Gamma^k_{ij,\epsilon} &= \bar{w}_{k,\epsilon}(\bar{\mathbf{g}}^{k,\epsilon}\cdot\bar{\partial}_i\bar{\mathbf{g}}_{j,\epsilon}) = \bar{\partial}_i(\bar{\mathbf{v}}_\epsilon\cdot\bar{\mathbf{g}}_{j,\epsilon}) - \bar{\partial}_i\bar{\mathbf{v}}_\epsilon\cdot\bar{\mathbf{g}}_{j,\epsilon}, \\
\bar{w}_{k,\epsilon}\Gamma^k_{ij,\epsilon} &= \bar{\partial}_j(\bar{\mathbf{v}}_\epsilon\cdot\bar{\mathbf{g}}_{i,\epsilon}) - \bar{\partial}_j\bar{\mathbf{v}}_\epsilon\cdot\bar{\mathbf{g}}_{i,\epsilon},
\end{aligned}$$

and the notation

$$\bar{\omega}^\epsilon_{ij}(\bar{\mathbf{v}}_\epsilon) := \frac{1}{2}\left(\bar{\partial}_i\bar{\mathbf{v}}_\epsilon\cdot\bar{\mathbf{g}}_{j,\epsilon} + \bar{\partial}_j\bar{\mathbf{v}}_\epsilon\cdot\bar{\mathbf{g}}_{i,\epsilon}\right),$$

that

$$D_{i||j}\bar{\mathbf{w}}_\epsilon = \bar{\omega}^\epsilon_{ij}(\bar{\mathbf{v}}_\epsilon).$$

Hence

$$D_{ij}\widetilde{\mathbf{v}} \circ \bar{\mathbf{P}}_\epsilon = \bar{\omega}^\epsilon_{kl}(\bar{\mathbf{v}}_\epsilon)[\bar{\mathbf{g}}^{k,\epsilon}]_i[\bar{\mathbf{g}}^{l,\epsilon}]_j.$$

If we want to pass to referential domain Ω, we have to use the mapping $\mathbf{R}_\epsilon$ (see (4.2.2)) and the chain rule to get (5.1.5).

Using the transformation we get (see [44]) that

$$\widetilde{A}^{ijkl} := \lambda\delta^{ij}\delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})$$

is replaced by $\bar{A}_\epsilon^{ijkl}$ and A_ϵ^{ijkl}, where

$$\bar{A}_\epsilon^{ijkl} := \lambda\bar{g}^{ij,\epsilon}\bar{g}^{kl,\epsilon} + \mu(\bar{g}^{ik,\epsilon}\bar{g}^{jl,\epsilon} + \bar{g}^{il,\epsilon}\bar{g}^{jk,\epsilon}),\ A_\epsilon^{ijkl}(x) := \bar{A}_\epsilon^{ijkl}(\mathbf{R}_\epsilon(x)). \quad (5.1.9)$$

To avoid confusion with tildes and bars let us also use the notation

$$A_0^{ijkl} := \lambda\delta^{ij}\delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}). \quad (5.1.10)$$

Similarly we can derive

$$d\widetilde{S}_\epsilon d\widetilde{y}_1 = o_\epsilon\sqrt{\nu_i o^{ij,\epsilon}\nu_j}dSdx_1 = \epsilon^2 d_\epsilon\sqrt{\nu_i o^{ij,\epsilon}\nu_j}dSdx_1$$

(see [44] and (4.2.16)).

At the end of this subsection, we introduce the limit model, formulate assumptions and state our main result. Let us denote

$$\check{\mathbf{f}}_{\mathbf{f}+\mathbf{h}}(x_1,t) \ := \ \int_S \mathbf{f}(x_1,x_2,x_3,t)\ dx_2dx_3 + \int_{\partial S}\mathbf{h}(x_1,x_2,x_3,t)\ dS \quad (5.1.11)$$

$$\check{\rho}(x_1) \ := \ \int_S \rho(x_1,x_2,x_3)\ dx_2dx_3, \quad (5.1.12)$$

for $(x_1,t) \in (0,l)\times(0,T,)$ and $x_1 \in (0,l)$, respectively. We show how to prove that after a suitable limit process we come to the asymptotic dynamic model

$$-\int_0^T \dot{\varphi}(t)\int_0^l \check{\rho}\partial_t\mathbf{u}(t)\cdot\mathbf{v}\ dx_1dt + \int_0^T\varphi(t)\int_0^l E[I_1(\partial_1\mathbf{u}_*(t)\cdot\mathbf{b})(\mathbf{v}_*'\cdot\mathbf{b})+$$

$$+I_2(\partial_1\mathbf{u}_*(t)\cdot\mathbf{n})(\mathbf{v}_*'\cdot\mathbf{n})]\ dx_1dt + \int_0^T\varphi(t)\int_0^l \mu K(\partial_1\mathbf{u}_*(t)\cdot\mathbf{t})(\mathbf{v}_*'\cdot\mathbf{t})\ dx_1dt =$$

$$= \int_0^T\varphi(t)\int_0^l(\check{\mathbf{f}}_{\mathbf{f}+\mathbf{h}}(t)\cdot\mathbf{v})\ dx_1dt \quad (5.1.13)$$

for all functions $\varphi \in C_0^\infty([0,T])$ and $\mathbf{v}_* \in W_0^{1,2}(0,l)^3$ generated by an arbitrary couple $\langle\mathbf{v},\psi\rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$. To shorten the notation we omit variable x_1 from (5.1.13). The function $\mathbf{u}$ that, together with the function ϕ, generates the function $\mathbf{u}_*$ (see (4.2.17)), satisfies the initial state

$$\mathbf{u}(x_1,0) = \mathbf{q}_0(x_1) \text{ and } (\check{\rho}\partial_t\mathbf{u})(x_1,0) = (\check{\rho}\mathbf{q}_1)(x_1),\ x_1 \in (0,l). \quad (5.1.14)$$

To prove any relation between (5.1.3)–(5.1.4) and (5.1.13)–(5.1.14) we must define suitable scaling and assumptions. Let us assume that

1. $\rho_\epsilon = \epsilon^2\rho$, where $\rho \in L^\infty(\Omega)$ and

$$0 < C_0 \le \rho \le C_1 \text{ a.e. in } \Omega; \quad (5.1.15)$$

2. $\mathbf{f}_\epsilon = \epsilon^2\mathbf{f}$, $\mathbf{f} \in L^2(0,T;L^2(\Omega)^3)$, $\mathbf{h}_\epsilon = \epsilon^3\mathbf{h}$, $\mathbf{h} \in W^{1,1}(0,T;L^2(0,l;L^2(\partial S)^3))$;

3. $\{\mathbf{q}_{0,\epsilon}\}_{\epsilon\in(0,1)} \subset V(\Omega)$, $\{\mathbf{q}_{1,\epsilon}\}_{\epsilon\in(0,1)} \subset L^2(\Omega)^3$,

$$\frac{1}{\epsilon}\|\omega^\epsilon(\mathbf{q}_{0,\epsilon})\|_2 \leq C, \ \forall \epsilon \in (0,1), \tag{5.1.16}$$

where the constant C is independent of ϵ, and

$$\mathbf{q}_{0,\epsilon} \rightharpoonup \mathbf{q}_0 \text{ in } V(\Omega), \ \mathbf{q}_{1,\epsilon} \rightharpoonup \mathbf{q}_1 \text{ in } L^2(\Omega)^3 \tag{5.1.17}$$

for $\epsilon \to 0$, where $\mathbf{q}_0 \in W_0^{1,2}(0,l)^3$ and $\mathbf{q}_1 \in L^2(0,l)^3$, i.e. the limit functions $\mathbf{q}_0$, $\mathbf{q}_1$ are the constant functions in the second and third variable.

The reason for the choice of scaling is that without it we are unable to derive suitable inequalities and consequently a priori estimates for $\mathbf{u}_\epsilon$ (see (4.1.67), (5.1.87) and (5.1.88)). Without the estimates, we are not able to guarantee boundedness of the functions $\mathbf{u}_\epsilon$ in appropriate spaces, which means that the curved rods can be broken when the diameter converges to zero.

After substitution of the above assumptions to (5.1.3)–(5.1.4), we get

$$\int_0^T \int_\Omega [-\rho\partial_t \mathbf{u}_\epsilon(t)\cdot\partial_t\mathbf{v}(t) + A_\epsilon^{ijkl}\frac{1}{\epsilon}\omega_{kl}^\epsilon(\mathbf{u}_\epsilon(t))\frac{1}{\epsilon}\omega_{ij}^\epsilon(\mathbf{v}(t))]\, d_\epsilon\, dxdt =$$

$$= \int_0^T \int_\Omega \mathbf{f}(t)\cdot\mathbf{v}(t)\, d_\epsilon\, dxdt + \int_0^T\int_0^l\int_{\partial S} \mathbf{h}(t)\cdot\mathbf{v}(t)\, d_\epsilon \epsilon\sqrt{\nu_i o^{ij,\epsilon}\nu_j}\, dSdx_1dt \tag{5.1.18}$$

for all $\mathbf{v} \in C_0^\infty(0,T;V(\Omega))$, and

$$\mathbf{u}_\epsilon(x_1,0) = \mathbf{q}_{0,\epsilon}(x_1), \ \partial_t\mathbf{u}_\epsilon(x_1,0) = \mathbf{q}_{1,\epsilon}(x_1), \ x_1 \in (0,l). \tag{5.1.19}$$

We finish the section with our main result.

Theorem 5.1.1 *Let a function $\mathbf{\Phi} \in W^{1,\infty}(0,l)^3$ be a parametrization of a unit speed curve. Let $\mathbf{f} \in L^2(0,T;L^2(\Omega)^3)$, $\mathbf{h} \in W^{1,1}(0,T;L^2(0,l;L^2(\partial S)^3))$ and $\check{\mathbf{f}}_{\mathbf{f}+\mathbf{h}}$ be defined in* (5.1.11). *Then, there is a unique pair $\langle \mathbf{u},\phi\rangle \in L^\infty(0,T;\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l))$ such that $\partial_t\mathbf{u} \in L^\infty(0,T;L^2(0,l)^3)\cap C([0,T];[\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)]')$ that generates a unique solution to the problem* (5.1.13)–(5.1.14)*. Moreover, the constant extension to $\Omega = (0,l)\times S$ of $\langle\mathbf{u},\phi\rangle$ may be approximated by solutions $\mathbf{u}_\epsilon \in L^\infty(0,T;V(\Omega))\cap W^{1,\infty}(0,T;L^2(\Omega)^3)$ of the problem* (5.1.18)–(5.1.19) *as follows*

$$\begin{aligned} \mathbf{u} &= \lim_{\epsilon\to 0}\mathbf{u}_\epsilon \ *\text{-weakly in } L^\infty(0,T;V(\Omega)), \\ \partial_t\mathbf{u} &= \lim_{\epsilon\to 0}\partial_t\mathbf{u}_\epsilon \ *\text{-weakly in } L^\infty(0,T;L^2(\Omega)^3), \\ \phi &= \lim_{\epsilon\to 0}\frac{1}{2\epsilon}(\partial_2\mathbf{u}_\epsilon\cdot\mathbf{b}_\epsilon - \partial_3\mathbf{u}_\epsilon\cdot\mathbf{n}_\epsilon) \ *\text{-weakly in } L^\infty(0,T;L^2(\Omega)). \end{aligned}$$

To make the theorem shorter and less complex we did not include all assumptions and we postponed them to the next section.

5.1.2 Auxiliary propositions

We summarize now auxiliary propositions from [199] we need for the proof of the main result. We also mention one useful proposition from [100].

Proposition 5.1.2 *Let $\lambda \geq 0$, $\mu > 0$ and A_ϵ^{ijkl} be defined in* (5.1.9)*, i.e.*

$$A_\epsilon^{ijkl} = \lambda g^{ij,\epsilon}g^{kl,\epsilon} + \mu(g^{ik,\epsilon}g^{jl,\epsilon} + g^{il,\epsilon}g^{jk,\epsilon}).$$

Then there exists a constant $C > 0$ such that the estimate

$$\sum_{i,j=1}^{3} |t_{ij}|^2 \leq C A_\epsilon^{ijkl}(x) t_{kl} t_{ij} \tag{5.1.20}$$

holds for all $x \in \overline{\Omega}$, all $\epsilon \in [0,1]$ and all symmetric matrices $(t_{ij})_{i,j=1}^3$, with the constant C being independent of ϵ and x.

Proof: First, we verify that

$$g^{ik,\epsilon}(x) g^{jl,\epsilon}(x) t_{kl} t_{ij} > 0 \text{ if } t_{ij} \neq 0$$

for all $\epsilon \in [0,1]$ and $x \in \overline{\Omega}$. In the case of $\epsilon \in (0,1]$, the proof proceeds in the same way as in [44] Theorem 1.8-1.The case $\epsilon = 0$ is an obvious consequence of (4.2.7), (4.2.9), (4.2.10), Proposition 4.3.2, (4.3.6) and (5.1.10). The mapping

$$(\epsilon, x, (t_{ij})) \in \mathbf{K} := [0,1] \times \overline{\Omega} \times \{t_{ij};\ \sum_{i,j=1}^{3} |t_{ij}|^2 = 1\} \to g^{ik,\epsilon}(x) g^{jl,\epsilon}(x) t_{kl} t_{ij}$$

is continuous. The only difficulty could appear for $\epsilon \to 0$. The domain $\mathbf{K}$ is however compact and, for instance, the terms

$$g^{12,\epsilon}(x) = -\frac{\epsilon x_3 \gamma_\epsilon(x_1)}{d_\epsilon^2(x)},\ \forall x \in \overline{\Omega},$$

converge to zero in $C(\overline{\Omega})$ for $\epsilon \to 0$ because of (4.2.7)–(4.2.10) and Corollary 4.3.3. We thus infer

$$C = \inf_{(\epsilon, x, (t_{ij})) \in \mathbf{K}} g^{ik,\epsilon}(x) g^{jl,\epsilon}(x) t_{kl} t_{ij} > 0.$$

Hence, we get the assertion of the proposition. □

Proposition 5.1.3 [100] *Let $w \in W^{1,2}(\Omega)$. Then $\partial_i \partial_j w \in L^2(0,l;[W_0^{1,2}(S)]')$ for $i,\ j = 1,2,3$ except for $i = j = 1$. If, in addition, $w|_{x_1=0} = w|_{x_1=l} = 0$, then $\partial_j w|_{x_1=0} = \partial_j w|_{x_1=l} = 0$, for $j = 2,3$, in the sense of the space $C([0,l];[W_0^{1,2}(S)]')$. Furthermore, if $v \in L^2(0,l;L^2(S))$, $\partial_1 v \in L^2(0,l;[W_0^{1,2}(S)]')$ and $v|_{x_1=0} = v|_{x_1=l} = 0$ in the sense of the space $C([0,l];[W_0^{1,2}(S)]')$, then there is a constant C independent of v such that*

$$\|v\|_{L^2(0,l;L^2(S))} \leq C \|\nabla v\|_{L^2(0,l;[W_0^{1,2}(S)]')}. \tag{5.1.21}$$

Proposition 5.1.4 [100] *Let sequences $\{v_n\}_{n=1}^\infty \subset L^2(0,l;L^2(S))$ and $\{\partial_1 v_n\}_{n=1}^\infty \subset L^2(0,l;[W_0^{1,2}(S)]')$ be such that $v_n|_{x_1=0} = v_n|_{x_1=l} = 0$, for all $n \in \mathbb{N}$, in the sense of the space $C([0,l];[W_0^{1,2}(S)]')$. Assume, in addition, that these sequences satisfy*

$$\partial_1 v_n \rightharpoonup \xi,\ \partial_j v_n \rightharpoonup 0 \text{ in } L^2(0,l;[W_0^{1,2}(S)]'),\ j = 2,3, \tag{5.1.22}$$

where $\xi \in L^2(0,l;[W_0^{1,2}(S)]')$. Then $\xi \in L^2(0,l)$ and there exists a unique function $v \in W_0^{1,2}(0,l)$ such that $v' = \xi$ and

$$v_n \rightharpoonup v \text{ in } L^2(0,l;L^2(S)), \tag{5.1.23}$$

$$v_n \to v \text{ in } C([0,l];[W^{1,2}(S)]'). \tag{5.1.24}$$

If the convergences in (5.1.22) *are strong then the convergence* (5.1.23) *is also strong.*

The next proposition follows from (4.2.15) and Corollary 4.3.3.

Proposition 5.1.5 *We have*

$$d_\epsilon \rightarrow 1 \text{ in } C(\overline{\Omega}), \tag{5.1.25}$$

$$\epsilon d_\epsilon(x)\sqrt{\nu_i(x)o^{ij,\epsilon}(x)\nu_j(x)} \rightarrow 1 \text{ in } C(\overline{(0,l)\times\partial S}), \tag{5.1.26}$$

for $\epsilon \rightarrow 0$, *where* ν_i, $i = 1,2,3$, *are components of a unit outward normal to* $(0,l)\times\partial S$. *Thus, there exist constants* C_j, $j = 0,1,2$, *such that* $0 < C_0 \leq d_\epsilon(x) \leq C_1$ *for all* $x \in \overline{\Omega}$, *and* $0 \leq d_\epsilon(x)\epsilon\sqrt{\nu_i(x)o^{ij,\epsilon}(x)\nu_j(x)} \leq C_2$ *for all* $x \in \overline{(0,l)\times\partial S}$ *and* $\epsilon \in (0,1)$.

5.1.3 Korn's inequality and stress tensor limit

This section deals with a version of Korn's inequality on curved thin domains and its consequences. For the kind of domains, it is necessary to identify the dependence of a constant from the inequality on the thickness of the domains, i.e. on ϵ. The following proposition is key to the version of Korn's inequality.

Proposition 5.1.6 *Suppose that* $\{\epsilon_n\}_{n=1}^{\infty} \subset (0,1)$ *and* $\epsilon_n \rightarrow 0$. *Let, in addition, a sequence* $\{\mathbf{u}_{\epsilon_n}\}_{n=1}^{\infty} \subset V(\Omega)$ *be such that*

$$\mathbf{u}_{\epsilon_n} \rightharpoonup \mathbf{u} \text{ in } W^{1,2}(\Omega)^3, \tag{5.1.27}$$

$$\frac{1}{\epsilon_n}\omega^{\epsilon_n}(\mathbf{u}_{\epsilon_n}) \rightharpoonup \zeta \text{ in } L^2(\Omega)^9 \tag{5.1.28}$$

for $\epsilon_n \rightarrow 0$. *Then the couple* $\langle \mathbf{u}, \phi\rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ *(in the sense* $\partial_j \mathbf{u} = 0$, $j = 2,3$*), where the function* ϕ *is such that*

$$\frac{1}{2\epsilon_n}(\partial_2\mathbf{u}_{\epsilon_n}\cdot\mathbf{b}_{\epsilon_n} - \partial_3\mathbf{u}_{\epsilon_n}\cdot\mathbf{n}_{\epsilon_n}) \rightharpoonup \phi \tag{5.1.29}$$

in $L^2(\Omega)$ *for* $\epsilon_n \rightarrow 0$. *In addition, the couple* $\langle \mathbf{u}, \phi\rangle$ *generates the function* $\mathbf{u}_* \in W_0^{1,2}(0,l)^3$ *(see* (4.2.17)*), which together with the function* $\mathbf{u}$ *satisfy the relations*

$$\mathbf{u}'\cdot\mathbf{t} = 0 \text{ a.e. on } [0,l], \tag{5.1.30}$$

$$\mathbf{u}_*'\cdot\mathbf{t} = \partial_3\zeta_{12} - \partial_2\zeta_{13} \text{ in } L^2(0,l;[W_0^{1,2}(S)]'), \tag{5.1.31}$$

$$\mathbf{u}_*'\cdot\mathbf{n} = -\partial_3\zeta_{11} \text{ a.e. on } [0,l], \tag{5.1.32}$$

$$\mathbf{u}_*'\cdot\mathbf{b} = \partial_2\zeta_{11} \text{ a.e. on } [0,l]. \tag{5.1.33}$$

If the sequence $\{\frac{1}{\epsilon_n}\omega^{\epsilon_n}(\mathbf{u}_{\epsilon_n})\}_{n=1}^{\infty}$ *converges strongly in* $L^2(\Omega)^9$, *then the convergence in* (5.1.27) *is strong as well.*

Proof: The proof is a shorter version of the proof of Proposition 7.2 from [199]. We decompose the proof into several steps.

1. *We show the function* $\mathbf{u}$ *depends only on* x_1.

   Since $\mathbf{u}_\epsilon \in V(\Omega)$, $\forall\epsilon \in (0,1)$, the convergence (5.1.27) implies that the function $\mathbf{u} \in V(\Omega)$ as well. Function $\mathbf{u}$ can be expressed as

   $$\mathbf{u} = (\mathbf{u}\cdot\mathbf{t})\mathbf{t} + (\mathbf{u}\cdot\mathbf{n})\mathbf{n} + (\mathbf{u}\cdot\mathbf{b})\mathbf{b}.$$

   It is thus enough to check that $\mathbf{u}\cdot\mathbf{t}$, $\mathbf{u}\cdot\mathbf{n}$ and $\mathbf{u}\cdot\mathbf{b}$ depends on x_1.

   From (5.1.27), (5.1.28) together with (5.1.5)–(5.1.8) we get

   $$\partial_i\mathbf{u}_{\epsilon_n}\cdot\mathbf{g}_{1,\epsilon_n} \rightarrow \partial_i\mathbf{u}\cdot\mathbf{t} = 0 \text{ in } L^2(\Omega),\ i = 2,3.$$

$\mathbf{u} \cdot \mathbf{t}$ thus depends only on x_1. Using the same approach we can deduce

$$\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n} \to \partial_2 \mathbf{u} \cdot \mathbf{n} = 0,\ \partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n} \to \partial_3 \mathbf{u} \cdot \mathbf{b} = 0,$$

and

$$(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n} + \partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}) \to (\partial_2 \mathbf{u} \cdot \mathbf{b} + \partial_3 \mathbf{u} \cdot \mathbf{n}) = 0.$$

Using the first relations we arrive at

$$(\mathbf{u} \cdot \mathbf{n})(x_1, x_2, x_3) = \widehat{\xi}_1(x_1, x_3) \text{ and } (\mathbf{u}, \mathbf{b})(x_1, x_2, x_3) = \widehat{\xi}_2(x_1, x_2),$$

where $\widehat{\xi}_i \in L^\infty(0, l; L^2(S)) \cap L^2(0, l; W^{1,2}(S))$, $i = 1, 2$. Integrating the second relation over rectangle $[x_2^0, x_2] \times [x_3^0, x_3]$ we get

$$\Big(\widehat{\xi}_1(x_1, x_3) - \widehat{\xi}_1(x_1, x_3^0)\Big)x_2 = -\Big(\widehat{\xi}_2(x_1, x_2) - \widehat{\xi}_2(x_1, x_2^0)\Big)x_3.$$

After integration over a second rectangle with nonempty intersection we can check that there is no dependence on point (x_2^0, x_3^0). We thus have

$$\mathbf{u} \cdot \mathbf{n} = \xi_1(x_1)x_3 + \xi_2(x_1),\ \mathbf{u} \cdot \mathbf{b} = -\xi_1(x_1)x_2 + \xi_3(x_1) \text{ in } \Omega. \tag{5.1.34}$$

Since, in addition, the functions $\mathbf{n}$ and $\mathbf{b} \in L^\infty(0, l)^3$, the functions $\xi_i \in L^2(0, l)$, $i = 1, 2, 3$.

It remains to prove that $\xi_1(x_1) = 0$. From (5.1.34) we have

$$\partial_1(\partial_3 \mathbf{u} \cdot \mathbf{n}) - \partial_1(\partial_2 \mathbf{u} \cdot \mathbf{b}) = 2\xi_1' \text{ in } [W_0^{1,2}(\Omega)]'. \tag{5.1.35}$$

In an analogous way to the above, we can check that $\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n} - \partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}$ converges to $\partial_3 \mathbf{u} \cdot \mathbf{n} - \partial_2 \mathbf{u} \cdot \mathbf{b}$ weakly in $L^2(\Omega)$ and thus

$$\partial_1(\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}) - \partial_1(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}) \rightharpoonup \partial_1(\partial_3 \mathbf{u} \cdot \mathbf{n}) - \partial_1(\partial_2 \mathbf{u} \cdot \mathbf{b}) \text{ in } [W_0^{1,2}(\Omega)]'.$$

Changing the positions of derivatives we get the identities

$$\frac{1}{\epsilon_n}\Big(\partial_3(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{g}_{1,\epsilon_n}) - \partial_2(\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{g}_{1,\epsilon_n})\Big) =$$

$$= \beta_{\epsilon_n}(\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n}) - \alpha_{\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n}) + \gamma_{\epsilon_n}(\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n} + \partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}) \tag{5.1.36}$$

and

$$\partial_3(\partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}) - \partial_2(\partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}) = \Big(\partial_1(\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}) - \partial_1(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n})\Big) +$$

$$+ \beta_{\epsilon_n}(\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n}) - \alpha_{\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n}) + \gamma_{\epsilon_n}(\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n} + \partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}) \tag{5.1.37}$$

in $[W_0^{1,2}(\Omega)]'$. If we subtract identities (5.1.36) and (5.1.37) and use again (5.1.27), (5.1.28) together with (5.1.5)–(5.1.8), we arrive at

$$\partial_1(\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}) - \partial_1(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}) \to 0 \text{ in } L^2(0, l; [W_0^{1,2}(\Omega)]').$$

Hence and from (5.1.35) we have $\xi_1' = 0$ in the sense of distributions and thus ξ_1 is a constant.

As the last step we prove that $\xi_1 = 0$. Since we know properties of $\mathbf{u} \cdot \mathbf{t}$, $\mathbf{u} \cdot \mathbf{n}$ and $\mathbf{u} \cdot \mathbf{b}$ we can write

$$\begin{aligned} \mathbf{u}(x_1, x_2, x_3) &= (\mathbf{u} \cdot \mathbf{t})\mathbf{t}(x_1) + (\xi_1 x_3 + \xi_2(x_1))\mathbf{n}(x_1) + \\ &+ (-\xi_1 x_2 + \xi_3(x_1))\mathbf{b}(x_1). \end{aligned} \tag{5.1.38}$$

Due to $\mathbf{u} \in V(\Omega)$ we know that $\partial_j \mathbf{u} \in C_0(0,l;[W_0^{1,2}(S)^3]')$, $j = 2,3$ (see [100]). Taking $\varphi \in W_0^{1,2}(S)$ such that $\int_S \varphi \; dx_2dx_3 = 1$, we get from (5.1.38) that

$$\int_S \partial_2 \mathbf{u}(x_1,x_2,x_3)\varphi \; dx_2dx_3 = -\xi_1 \mathbf{b}(x_1), \; x_1 \in [0,l].$$

Hence we get that $-\xi_1 \mathbf{b} \in C_0(0,l)^3$. If $\mathbf{b}$ is continuous then

$$0 = \lim_{x_1 \to 0} \int_S \partial_2 \mathbf{u}(x_1)\varphi \; dx_2dx_3 = -\xi_1 \mathbf{b}(0).$$

But $|\mathbf{b}(0)| = 1$ and thus $\xi_1 = 0$. If $\mathbf{b}$ is not continuous in x_1 then$\xi_1 = 0$ as well.

2. *We prove the identity* (5.1.30).

   The relation is a direct consequence of (5.1.27), (5.1.28), (5.1.5), and (5.1.8).

3. *If we put*

$$\phi_{\epsilon_n} := \frac{1}{2\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n} - \partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}) \tag{5.1.39}$$

   then

$$\partial_j \phi_{\epsilon_n} \to 0 \; in \; L^2(0,l;[W_0^{1,2}(S)]'), \; j = 2,3, \tag{5.1.40}$$

   for $\epsilon_n \to 0$ *and* $\phi_{\epsilon_n}|_{x_1=0} = \phi_{\epsilon_n}|_{x_1=l} = 0$ *for* $\epsilon_n \in (0,1)$ *in the sense of the space* $C([0,l];[W_0^{1,2}(S)]')$.

   The result follows from Proposition 5.1.3 and application of (5.1.27), (5.1.28), (5.1.5)–(5.1.8) to the relation

$$\partial_2 \phi_{\epsilon_n} = \frac{1}{2\epsilon_n}\Big(\partial_2(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}) + \partial_2(\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n})\Big) - \frac{1}{\epsilon_n}\partial_3(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}),$$

   which holds in $L^2(0,l;[W_0^{1,2}(S)]')$. The same arguments can be applied to $\partial_3 \phi_{\epsilon_n}$.

4. *Let us define* $\mathbf{u}_{*,\epsilon_n} := (u_{*,1}^{\epsilon_n}, u_{*,2}^{\epsilon_n}, u_{*,3}^{\epsilon_n})$ *by*

$$u_{*,1}^{\epsilon_n} := -\phi_{\epsilon_n}, \; u_{*,2}^{\epsilon_n} := -\frac{1}{\epsilon_n}(\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{g}_{1,\epsilon_n}), \; u_{*,3}^{\epsilon_n} := \frac{1}{\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{g}_{1,\epsilon_n}),$$

   where

$$\mathbf{u}_{*,\epsilon_n} := -\phi_{\epsilon_n}\mathbf{t}_{\epsilon_n} - \frac{1}{\epsilon_n}(\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{g}_{1,\epsilon_n})\mathbf{n}_{\epsilon_n} + \frac{1}{\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{g}_{1,\epsilon_n})\mathbf{b}_{\epsilon_n}. \tag{5.1.41}$$

   As a result of our assumptions, we get

$$\begin{aligned} \partial_1 \mathbf{u}_{*,\epsilon_n} \cdot \mathbf{t}_{\epsilon_n} &\rightharpoonup \partial_3 \zeta_{12} - \partial_2 \zeta_{13} \; in \; L^2(0,l;[W_0^{1,2}(S)]'), && (5.1.42)\\ \partial_1 \mathbf{u}_{*,\epsilon_n} \cdot \mathbf{b}_{\epsilon_n} &\rightharpoonup \partial_2 \zeta_{11} \; in \; L^2(0,l;[W_0^{1,2}(S)]'), && (5.1.43)\\ \partial_1 \mathbf{u}_{*,\epsilon_n} \cdot \mathbf{n}_{\epsilon_n} &\rightharpoonup -\partial_3 \zeta_{11} \; in \; L^2(0,l;[W_0^{1,2}(S)]') && (5.1.44) \end{aligned}$$

   and thus

$$\partial_1 \mathbf{u}_{*,\epsilon_n} \rightharpoonup (\partial_3 \zeta_{12} - \partial_2 \zeta_{13})\mathbf{t} - \partial_3 \zeta_{11}\mathbf{n} + \partial_2 \zeta_{11}\mathbf{b} \tag{5.1.45}$$

   in $L^2(0,l;[W_0^{1,2}(S)^3]')$ *for* $\epsilon_n \to 0$.

   From (5.1.28) and (5.1.5)–(5.1.8), it follows that

$$\frac{1}{\epsilon_n^2}\partial_3 \theta_{12}^{\epsilon_n}(\mathbf{u}_{\epsilon_n}) + \frac{1}{\epsilon_n}\partial_3 \kappa_{12}^{\epsilon}(\mathbf{u}_{\epsilon_n}) - \frac{1}{\epsilon_n^2}\partial_2 \theta_{13}^{\epsilon_n}(\mathbf{u}_{\epsilon_n}) - \frac{1}{\epsilon_n}\partial_2 \kappa_{13}^{\epsilon_n}(\mathbf{u}_{\epsilon_n}) \rightharpoonup \partial_3 \zeta_{12} - \partial_2 \zeta_{13}$$

and

$$\frac{\partial_j \kappa_{11}^{\epsilon_n}(\mathbf{u}_{\epsilon_n})}{\epsilon_n} \rightharpoonup \partial_j \zeta_{11}, \ j = 2,3,$$

in $L^2(0,l;[W_0^{1,2}(S)]')$ for $\epsilon_n \to 0$. Thus to prove (5.1.42)–(5.1.45) it is enough to check that

$$\partial_1 \mathbf{u}_{*,\epsilon_n} \cdot \mathbf{t}_{\epsilon_n} - \Big(\frac{1}{\epsilon_n^2}\partial_3\theta_{12}^{\epsilon_n}(\mathbf{u}_{\epsilon_n}) + \frac{1}{\epsilon_n}\partial_3\kappa_{12}^{\epsilon_n}(\mathbf{u}_{\epsilon_n}) -$$

$$-\frac{1}{\epsilon_n^2}\partial_2\theta_{13}^{\epsilon_n}(\mathbf{u}_{\epsilon_n}) - \frac{1}{\epsilon_n}\partial_2\kappa_{13}^{\epsilon_n}(\mathbf{u}_{\epsilon_n})\Big) \to 0 \text{ in } L^2(\Omega), \quad (5.1.46)$$

$$\partial_1 \mathbf{u}_{*,\epsilon_n} \cdot \mathbf{b}_{\epsilon_n} - \frac{\partial_2\kappa_{11}^{\epsilon_n}(\mathbf{u}_{\epsilon_n})}{\epsilon_n} \to 0 \text{ in } L^2(\Omega), \quad (5.1.47)$$

$$\partial_1 \mathbf{u}_{*,\epsilon_n} \cdot \mathbf{n}_{\epsilon_n} + \frac{\partial_3\kappa_{11}^{\epsilon_n}(\mathbf{u}_{\epsilon_n})}{\epsilon_n} \to 0 \text{ in } L^2(\Omega). \quad (5.1.48)$$

First, we find the expressions for the terms $\partial_1 \mathbf{u}_{*,\epsilon_n} \cdot \mathbf{t}_{\epsilon_n}$, $\partial_1 \mathbf{u}_{*,\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}$, and $\partial_1 \mathbf{u}_{*,\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}$. Using the definitions (5.1.5)–(5.1.8) of the tensors θ^{ϵ_n} and κ^{ϵ_n}, it is easy to see that it is enough to add (5.1.36) to (5.1.37) and to multiply this sum with $\frac{1}{2\epsilon_n}$ to get

$$\frac{1}{\epsilon_n^2}\partial_3\theta_{12}^{\epsilon_n}(\mathbf{u}_{\epsilon_n}) + \frac{1}{\epsilon_n}\partial_3\kappa_{12}^{\epsilon_n}(\mathbf{u}_{\epsilon_n}) - \frac{1}{\epsilon_n^2}\partial_2\theta_{13}^{\epsilon_n}(\mathbf{u}_{\epsilon_n}) - \frac{1}{\epsilon_n}\partial_2\kappa_{13}^{\epsilon_n}(\mathbf{u}_{\epsilon_n}) =$$

$$= \frac{1}{2\epsilon_n}\Big(\partial_1(\partial_3\mathbf{u}_{\epsilon_n}\cdot\mathbf{n}_{\epsilon_n}) - \partial_1(\partial_2\mathbf{u}_{\epsilon_n}\cdot\mathbf{b}_{\epsilon_n})\Big) +$$

$$+\frac{1}{\epsilon_n}\Bigg(\beta_{\epsilon_n}(\partial_3\mathbf{u}_{\epsilon_n}\cdot\mathbf{t}_{\epsilon_n}) - \alpha_{\epsilon_n}(\partial_2\mathbf{u}_{\epsilon_n}\cdot\mathbf{t}_{\epsilon_n}) + \gamma_{\epsilon_n}\Big(\partial_3\mathbf{u}_{\epsilon_n}\cdot\mathbf{b}_{\epsilon_n} + \partial_2\mathbf{u}_{\epsilon_n}\cdot\mathbf{n}_{\epsilon_n}\Big)\Bigg).$$

By rewriting the above mentioned expression in such a way that it involves the terms $\frac{1}{\epsilon_n}\beta_{\epsilon_n}(\partial_3\mathbf{u}_{\epsilon_n}\cdot\mathbf{g}_{1,\epsilon_n})$ and $\frac{1}{\epsilon_n}\alpha_{\epsilon_n}(\partial_2\mathbf{u}_{\epsilon_n}\cdot\mathbf{g}_{1,\epsilon_n})$ instead of $\frac{1}{\epsilon_n}\beta_{\epsilon_n}(\partial_3\mathbf{u}_{\epsilon_n}\cdot\mathbf{t}_{\epsilon_n})$ and $\frac{1}{\epsilon_n}\alpha_{\epsilon_n}(\partial_2\mathbf{u}_{\epsilon_n}\cdot\mathbf{t}_{\epsilon_n})$, we conclude that

$$\frac{1}{\epsilon_n^2}\partial_3\theta_{12}^{\epsilon_n}(\mathbf{u}_{\epsilon_n}) + \frac{1}{\epsilon_n}\partial_3\kappa_{12}^{\epsilon_n}(\mathbf{u}_{\epsilon_n}) - \frac{1}{\epsilon_n^2}\partial_2\theta_{13}^{\epsilon_n}(\mathbf{u}_{\epsilon_n}) - \frac{1}{\epsilon_n}\partial_2\kappa_{13}^{\epsilon_n}(\mathbf{u}_{\epsilon_n}) =$$

$$= \Big(-\partial_1\phi_{\epsilon_n} + \frac{1}{\epsilon_n}\beta_{\epsilon_n}(\partial_3\mathbf{u}_{\epsilon_n}\cdot\mathbf{g}_{1,\epsilon_n}) - \frac{1}{\epsilon_n}\alpha_{\epsilon_n}(\partial_2\mathbf{u}_{\epsilon_n}\cdot\mathbf{g}_{1,\epsilon_n})\Big) +$$

$$+\Big((\beta_{\epsilon_n}^2 x_2 + \alpha_{\epsilon_n}\beta_{\epsilon_n}x_3)(\partial_3\mathbf{u}_{\epsilon_n}\cdot\mathbf{t}_{\epsilon_n}) - (\alpha_{\epsilon_n}\beta_{\epsilon_n}x_2 + \alpha_{\epsilon_n}^2 x_3)(\partial_2\mathbf{u}_{\epsilon_n}\cdot\mathbf{t}_{\epsilon_n})\Big) +$$

$$+\Bigg(\Big(\beta_{\epsilon_n}\gamma_{\epsilon_n}x_2 + \frac{\gamma_{\epsilon_n}}{\epsilon_n}\Big)(\partial_3\mathbf{u}_{\epsilon_n}\cdot\mathbf{b}_{\epsilon_n}) + \Big(\alpha_{\epsilon_n}\gamma_{\epsilon_n}x_3 + \frac{\gamma_{\epsilon_n}}{\epsilon_n}\Big)(\partial_2\mathbf{u}_{\epsilon_n}\cdot\mathbf{n}_{\epsilon_n})\Bigg) -$$

$$-\Big(\beta_{\epsilon_n}\gamma_{\epsilon_n}x_3(\partial_3\mathbf{u}_{\epsilon_n}\cdot\mathbf{n}_{\epsilon_n}) + \alpha_{\epsilon_n}\gamma_{\epsilon_n}x_2(\partial_2\mathbf{u}_{\epsilon_n}\cdot\mathbf{b}_{\epsilon_n})\Big). \quad (5.1.49)$$

in $[W_0^{1,2}(\Omega)]'$. In addition, all terms except $\partial_1\phi_{\epsilon_n}$ belong to $L^2(0,l;[W_0^{1,2}(S)]')$ then $\partial_1\phi_{\epsilon_n} \in L^2(0,l;[W_0^{1,2}(S)]')$ as well. From the definition of $\mathbf{u}_{*,\epsilon_n}$ (5.1.41), (4.2.3) and (5.1.49), it follows that

$$\partial_1\mathbf{u}_{*,\epsilon_n}\cdot\mathbf{t}_{\epsilon_n} =$$

$$= \Big(\frac{1}{\epsilon_n^2}\partial_3\theta_{12}^{\epsilon_n}(\mathbf{u}_{\epsilon_n}) + \frac{1}{\epsilon_n}\partial_3\kappa_{12}^{\epsilon_n}(\mathbf{u}_{\epsilon_n}) - \frac{1}{\epsilon_n^2}\partial_2\theta_{13}^{\epsilon_n}(\mathbf{u}_{\epsilon_n}) - \frac{1}{\epsilon_n}\partial_2\kappa_{13}^{\epsilon_n}(\mathbf{u}_{\epsilon_n})\Big) -$$

$$-\Big((\beta_{\epsilon_n}^2 x_2 + \alpha_{\epsilon_n}\beta_{\epsilon_n} x_3)(\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n}) - (\alpha_{\epsilon_n}\beta_{\epsilon_n} x_2 + \alpha_{\epsilon_n}^2 x_3)(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n})\Big) -$$

$$-\left(\left(\beta_{\epsilon_n}\gamma_{\epsilon_n} x_2 + \frac{\gamma_{\epsilon_n}}{\epsilon_n}\right)(\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}) + \left(\alpha_{\epsilon_n}\gamma_{\epsilon_n} x_3 + \frac{\gamma_{\epsilon_n}}{\epsilon_n}\right)(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n})\right) +$$

$$+\Big(\beta_{\epsilon_n}\gamma_{\epsilon_n} x_3(\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}) + \alpha_{\epsilon_n}\gamma_{\epsilon_n} x_2(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n})\Big) \tag{5.1.50}$$

in $L^2(0,l;[W_0^{1,2}(S)]')$.

Further, using (4.2.6) and (5.1.8), we get (in the sense of $L^2(0,l;[W_0^{1,2}(S)]')$) that

$$\frac{\partial_2 \kappa_{11}^{\epsilon_n}(\mathbf{u}_{\epsilon_n})}{\epsilon_n} = \frac{1}{\epsilon_n}\partial_2(\partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n}) - \partial_2(\partial_1 \mathbf{u}_{\epsilon_n} \cdot (x_2 \beta_{\epsilon_n} \mathbf{t}_{\epsilon_n})) -$$

$$-\partial_2(\partial_1 \mathbf{u}_{\epsilon_n} \cdot (x_3 \alpha_{\epsilon_n} \mathbf{t}_{\epsilon_n})) + \partial_2(\partial_1 \mathbf{u}_{\epsilon_n} \cdot (x_3 \gamma_{\epsilon_n} \mathbf{n}_{\epsilon_n})) -$$

$$-\partial_2(\partial_1 \mathbf{u}_{\epsilon_n} \cdot (x_2 \gamma_{\epsilon_n} \mathbf{b}_{\epsilon_n})) = \sum_{j=1}^{5} I_j.$$

Now, we express the terms I_i, $i = 1, \dots, 5$, individually. Changing the position of the derivatives ∂_2 with ∂_1 in the terms above and using (4.2.3) and (4.2.6) lead (in the sense of the space $[W_0^{1,2}(\Omega)]'$) to

$$\begin{aligned} I_1 \;=\; & \frac{1}{\epsilon_n}\partial_1(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{g}_{1,\epsilon_n}) - \frac{\alpha_{\epsilon_n}}{\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}) - \frac{\beta_{\epsilon_n}}{\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}) + \\ & + x_2 \beta'_{\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n}) + x_2 \beta_{\epsilon_n}\partial_1(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n}) + x_3 \alpha'_{\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n}) + \\ & + x_3 \alpha_{\epsilon_n}\partial_1(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n}) - x_3 \gamma'_{\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}) - \\ & - x_3 \gamma_{\epsilon_n}\partial_1(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}) + x_2 \gamma'_{\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}) + x_2 \gamma_{\epsilon_n}\partial_1(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}), \end{aligned}$$

$$\begin{aligned} I_2 \;=\; & -\beta_{\epsilon_n}(\partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n}) - x_2 \beta_{\epsilon_n}\partial_1(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n}) + x_2 \alpha_{\epsilon_n}\beta_{\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}) + \\ & + x_2 \beta_{\epsilon_n}^2(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}), \end{aligned}$$

$$I_3 = -x_3 \alpha_{\epsilon_n}\partial_1(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n}) + x_3 \alpha_{\epsilon_n}^2(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}) + x_3 \alpha_{\epsilon_n}\beta_{\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}),$$

$$I_4 = x_3 \gamma_{\epsilon_n}\partial_1(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}) + x_3 \beta_{\epsilon_n}\gamma_{\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n}) + x_3 \gamma_{\epsilon_n}^2(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}),$$

$$\begin{aligned} I_5 \;=\; & -\gamma_{\epsilon_n}(\partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}) - x_2 \gamma_{\epsilon_n}\partial_1(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}) - x_2 \alpha_{\epsilon_n}\gamma_{\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n}) + \\ & + x_2 \gamma_{\epsilon_n}^2(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}). \end{aligned}$$

Then we get

$$\frac{\partial_2 \kappa_{11}^{\epsilon_n}(\mathbf{u}_{\epsilon_n})}{\epsilon_n} = \sum_{j=1}^{5} I_j = \frac{1}{\epsilon_n}\partial_1(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{g}_{1,\epsilon_n}) - \frac{\alpha_{\epsilon_n}}{\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}) -$$

$$-\frac{\beta_{\epsilon_n}}{\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}) + x_2 \beta'_{\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n}) + x_3 \alpha'_{\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n}) -$$

$$-x_3 \gamma'_{\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}) + x_2 \gamma'_{\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}) - \beta_{\epsilon_n}(\partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n}) +$$

$$+x_2 \alpha_{\epsilon_n}\beta_{\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}) + x_2 \beta_{\epsilon_n}^2(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}) + x_3 \alpha_{\epsilon_n}^2(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}) +$$

$$+x_3 \alpha_{\epsilon_n}\beta_{\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}) + x_3 \beta_{\epsilon_n}\gamma_{\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n}) + x_3 \gamma_{\epsilon_n}^2(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}) -$$

$$-\gamma_{\epsilon_n}(\partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}) - x_2 \alpha_{\epsilon_n}\gamma_{\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n}) + x_2 \gamma_{\epsilon_n}^2(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n})$$

in $[W_0^{1,2}(\Omega)]'$. In view of the definition of functions $\mathbf{u}_{*,\epsilon_n}$, we can derive (after rearrangement)

$$\partial_1 \mathbf{u}_{*,\epsilon_n} \cdot \mathbf{b}_{\epsilon_n} = \frac{1}{\epsilon_n}\partial_1(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{g}_{1,\epsilon_n}) - \alpha_{\epsilon_n}\phi_{\epsilon_n} + \gamma_{\epsilon_n}\frac{1}{\epsilon_n}(\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{g}_{1,\epsilon_n}) =$$

$$= \left(\frac{\partial_2 \kappa_{11}^{\epsilon_n}(\mathbf{u}_{\epsilon_n})}{\epsilon_n} + \frac{\alpha_{\epsilon_n}}{\epsilon_n}\left(\frac{(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}) + (\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n})}{2}\right)\right) +$$

$$+\gamma_{\epsilon_n}\left(\frac{1}{\epsilon_n}(\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{g}_{1,\epsilon_n}) + \partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}\right) -$$

$$-\left(\left(-\frac{\beta_{\epsilon_n}}{\epsilon_n} + \beta_{\epsilon_n}^2 x_2 + \alpha_{\epsilon_n}\beta_{\epsilon_n} x_3 - \gamma'_{\epsilon_n} x_3 + \gamma_{\epsilon_n}^2 x_2\right)(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n})\right) -$$

$$-\Big((\beta'_{\epsilon_n} x_2 + \alpha'_{\epsilon_n} x_3 + \beta_{\epsilon_n}\gamma_{\epsilon_n} x_3 - \alpha_{\epsilon_n}\gamma_{\epsilon_n} x_2)(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n})\Big) -$$

$$-\Big((\alpha_{\epsilon_n}\beta_{\epsilon_n} x_2 + \alpha_{\epsilon_n}^2 x_3 + \gamma_{\epsilon_n}^2 x_3 + \gamma'_{\epsilon_n} x_2)(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}) - \beta_{\epsilon_n}(\partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n})\Big) \quad (5.1.51)$$

in $L^2(0,l;[W_0^{1,2}(S)]')$. In an analogous way applied to $\frac{\partial_3 \kappa_{11}^{\epsilon_n}(\mathbf{u}_{\epsilon_n})}{\epsilon_n}$, we can derive that

$$\partial_1 \mathbf{u}_{*,\epsilon_n} \cdot \mathbf{n}_{\epsilon_n} = \left(-\frac{\partial_3 \kappa_{11}(\mathbf{u}_{\epsilon_n})}{\epsilon_n} + \frac{\beta_{\epsilon_n}}{\epsilon_n}\left(\frac{(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}) + (\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n})}{2}\right)\right) +$$

$$+\gamma_{\epsilon_n}\left(\frac{1}{\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{g}_{1,\epsilon_n}) + \partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}\right) +$$

$$+\left(\left(-\frac{\alpha_{\epsilon_n}}{\epsilon_n} + \alpha_{\epsilon_n}^2 x_3 + \alpha_{\epsilon_n}\beta_{\epsilon_n} x_2 + \gamma'_{\epsilon_n} x_2 + \gamma_{\epsilon_n}^2 x_3\right)(\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n})\right) +$$

$$+\Big((\beta'_{\epsilon_n} x_2 + \alpha'_{\epsilon_n} x_3 + \beta_{\epsilon_n}\gamma_{\epsilon_n} x_3 - \alpha_{\epsilon_n}\gamma_{\epsilon_n} x_2)(\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n})\Big) +$$

$$+\Big((\alpha_{\epsilon_n}\beta_{\epsilon_n} x_3 + \beta_{\epsilon_n}^2 x_2 + \gamma_{\epsilon_n}^2 x_2 - \gamma'_{\epsilon_n} x_3)(\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}) - \alpha_{\epsilon_n}(\partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n})\Big) \quad (5.1.52)$$

in $L^2(0,l;[W_0^{1,2}(S)]')$.

Now, we check the convergence (5.1.46). The convergences (5.1.47), (5.1.48) can be proved analogously. From (5.1.50) and the fact that $\mathbf{u}_{\epsilon_n} \in V(\Omega)$, α_{ϵ_n}, β_{ϵ_n}, $\gamma_{\epsilon_n} \in C^\infty([0,l])$, $\mathbf{g}_{1,\epsilon_n} \in C^\infty(\overline{\Omega})^3$, $\mathbf{t}_{\epsilon_n}$, $\mathbf{n}_{\epsilon_n}$, $\mathbf{b}_{\epsilon_n} \in C^\infty([0,l])^3$, it follows that the difference

$$\partial_1 \mathbf{u}_{*,\epsilon_n} \cdot \mathbf{t}_{\epsilon_n} - \left(\frac{\partial_3 \theta_{12}^{\epsilon_n}(\mathbf{u}_{\epsilon_n})}{\epsilon_n^2} + \frac{\partial_3 \kappa_{12}^{\epsilon_n}(\mathbf{u}_{\epsilon_n})}{\epsilon_n} - \frac{\partial_2 \theta_{13}^{\epsilon_n}(\mathbf{u}_{\epsilon_n})}{\epsilon_n^2} - \frac{\partial_2 \kappa_{13}^{\epsilon_n}(\mathbf{u}_{\epsilon_n})}{\epsilon_n}\right)$$

is well-defined in $L^2(\Omega)$ for $\epsilon_n \in (0,1)$ and satisfies due to (4.3.6) and (5.1.50) for $r \in \left(0, \frac{1}{3}\right)$ the estimate

$$\left\|\partial_1 \mathbf{u}_{*,\epsilon_n} \cdot \mathbf{t}_{\epsilon_n} - \left(\frac{\partial_3 \theta_{12}^{\epsilon_n}(\mathbf{u}_{\epsilon_n})}{\epsilon_n^2} + \frac{\partial_3 \kappa_{12}^{\epsilon_n}(\mathbf{u}_{\epsilon_n})}{\epsilon_n} - \frac{\partial_2 \theta_{13}^{\epsilon_n}(\mathbf{u}_{\epsilon_n})}{\epsilon_n^2} - \frac{\partial_2 \kappa_{13}^{\epsilon_n}(\mathbf{u}_{\epsilon_n})}{\epsilon_n}\right)\right\|_2$$

$$\leq C\left(\frac{1}{\epsilon_n^{2r}}\|\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n}\|_2 + \frac{1}{\epsilon_n^{2r}}\|\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n}\|_2 + \frac{1}{\epsilon_n^{1+r}}\|\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}\|_2 + \right.$$

$$\left. + \frac{1}{\epsilon_n^{1+r}}\|\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}\|_2 + \frac{1}{\epsilon_n^{2r}}\|\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}\|_2 + \frac{1}{\epsilon_n^{2r}}\|\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}\|_2\right) =$$

$$= C(\epsilon_n) + \frac{1}{\epsilon_n^{2r}}\Big(\|\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}\|_2 + \|\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}\|_2\Big), \quad (5.1.53)$$

where $C(\epsilon_n) \to 0$ for $\epsilon_n \to 0$ as a consequence of (5.1.5)–(5.1.8) and (5.1.28). It remains to study the behaviour of the terms

$$\frac{1}{\epsilon_n^{2r}}\|\partial_3\mathbf{u}_{\epsilon_n}\cdot\mathbf{n}_{\epsilon_n}\|_2,\ \frac{1}{\epsilon_n^{2r}}\|\partial_2\mathbf{u}_{\epsilon_n}\cdot\mathbf{b}_{\epsilon_n}\|_2.$$

The estimate

$$\left\|\frac{\partial_3\mathbf{u}_{\epsilon_n}\cdot\mathbf{n}_{\epsilon_n}}{\epsilon_n^{2r}}\right\|_2+\left\|\frac{\partial_2\mathbf{u}_{\epsilon_n}\cdot\mathbf{b}_{\epsilon_n}}{\epsilon_n^{2r}}\right\|_2 \le 2\left\|\frac{1}{\epsilon_n^{2r}}(\partial_3\mathbf{u}_{\epsilon_n}\cdot\mathbf{n}_{\epsilon_n}+\partial_2\mathbf{u}_{\epsilon_n}\cdot\mathbf{b}_{\epsilon_n})\right\|_2+$$

$$+2\left\|\frac{1}{\epsilon_n^{2r}}(\partial_3\mathbf{u}_{\epsilon_n}\cdot\mathbf{n}_{\epsilon_n}-\partial_2\mathbf{u}_{\epsilon_n}\cdot\mathbf{b}_{\epsilon_n})\right\|_2 \overset{(5.1.39)}{=} C_1(\epsilon_n)+4\epsilon_n^{1-2r}\|\phi_{\epsilon_n}\|_2 \overset{(5.1.21)}{\le}$$

$$\le C_1(\epsilon_n)+C\epsilon_n^{1-2r}\sum_{j=1}^{3}\|\partial_j\phi_{\epsilon_n}\|_{L^2(0,l;[W_0^{1,2}(S)]')} \overset{(5.1.40)}{=}$$

$$= C_1(\epsilon_n)+C_2(\epsilon_n)+C\epsilon_n^{1-2r}\|\partial_1\phi_{\epsilon_n}\|_{L^2(0,l;[W_0^{1,2}(S)]')} \overset{(5.1.49),(4.3.6)}{\le}$$

$$\le C_1(\epsilon_n)+C_2(\epsilon_n)+\frac{C}{\epsilon_n^{2r}}\left(\left\|\frac{1}{\epsilon_n}\partial_3\theta_{12}^{\epsilon_n}(\mathbf{u}_{\epsilon_n})+\partial_3\kappa_{12}^{\epsilon_n}(\mathbf{u}_{\epsilon_n})\right\|_{L^2(0,l;[W_0^{1,2}(S)]')}+\right.$$

$$\left.+\left\|\frac{1}{\epsilon_n}\partial_2\theta_{13}^{\epsilon_n}(\mathbf{u}_{\epsilon_n})+\partial_2\kappa_{13}^{\epsilon_n}(\mathbf{u}_{\epsilon_n})\right\|_{L^2(0,l;[W_0^{1,2}(S)]')}\right)+$$

$$+\frac{C}{\epsilon_n^{3r}}\Big(\|\partial_3\mathbf{u}_{\epsilon_n}\cdot\mathbf{g}_{1,\epsilon_n}\|_2+\|\partial_2\mathbf{u}_{\epsilon_n}\cdot\mathbf{g}_{1,\epsilon_n}\|_2\Big)+$$

$$+C{\epsilon_n}^{1-4r}\Big(\|\partial_3\mathbf{u}_{\epsilon_n}\cdot\mathbf{t}_{\epsilon_n}\|_2+\|\partial_2\mathbf{u}_{\epsilon_n}\cdot\mathbf{t}_{\epsilon_n}\|_2\Big)+$$

$$+\frac{C}{\epsilon_n^{3r}}\Big(\|\partial_3\mathbf{u}_{\epsilon_n}\cdot\mathbf{b}_{\epsilon_n}\|_2+\|\partial_2\mathbf{u}_{\epsilon_n}\cdot\mathbf{n}_{\epsilon_n}\|_2\Big)+$$

$$+C\epsilon_n^{1-2r}\left(\left\|\frac{1}{\epsilon_n^{2r}}(\partial_2\mathbf{u}_{\epsilon_n}\cdot\mathbf{b}_{\epsilon_n})\right\|_2+\left\|\frac{1}{\epsilon_n^{2r}}(\partial_3\mathbf{u}_{\epsilon_n}\cdot\mathbf{n}_{\epsilon_n})\right\|_2\right)=$$

$$=\sum_{j=1}^{6}C_j(\epsilon_n)+C\epsilon_n^{1-2r}\left(\left\|\frac{1}{\epsilon_n^{2r}}(\partial_2\mathbf{u}_{\epsilon_n}\cdot\mathbf{b}_{\epsilon_n})\right\|_2+\left\|\frac{1}{\epsilon_n^{2r}}(\partial_3\mathbf{u}_{\epsilon_n}\cdot\mathbf{n}_{\epsilon_n})\right\|_2\right)$$

can be rewritten as

$$\left\|\frac{1}{\epsilon_n^{2r}}(\partial_3\mathbf{u}_{\epsilon_n}\cdot\mathbf{n}_{\epsilon_n})\right\|_2+\left\|\frac{1}{\epsilon_n^{2r}}(\partial_2\mathbf{u}_{\epsilon_n}\cdot\mathbf{b}_{\epsilon_n})\right\|_2\le C\sum_{j=1}^{6}C_j(\epsilon_n).$$

We show now that $C_j(\epsilon_n)\to 0$, $j=1,\dots,6$, for $\epsilon_n\to 0$. $C_1(\epsilon_n)\to 0$ as a consequence of (5.1.28), (5.1.5), and (5.1.7) and thus

$$\frac{1}{\epsilon_n^{q_1}}(\partial_2\mathbf{u}_{\epsilon_n}\cdot\mathbf{b}_{\epsilon_n})+\partial_3\mathbf{u}_{\epsilon_n}\cdot\mathbf{n}_{\epsilon_n}\to 0$$

in $L^2(\Omega)$ for $q_1\in[0,2)$. From (5.1.27), (5.1.28), and (5.1.5)–(5.1.8) we can further derive the convergences

$$\begin{aligned}\partial_j\frac{1}{\epsilon_n^{q}}\left(\frac{1}{\epsilon_n}(\partial_2\mathbf{u}_{\epsilon_n}\cdot\mathbf{g}_{1,\epsilon_n})+\partial_1\mathbf{u}_{\epsilon_n}\cdot\mathbf{n}_{\epsilon_n}\right) &\to 0,\ j=2,3,\\ \partial_j\frac{1}{\epsilon_n^{q}}\left(\frac{1}{\epsilon_n}(\partial_3\mathbf{u}_{\epsilon_n}\cdot\mathbf{g}_{1,\epsilon_n})+\partial_1\mathbf{u}_{\epsilon_n}\cdot\mathbf{b}_{\epsilon_n}\right) &\to 0,\ j=2,3,\end{aligned}$$

in $L^2(0,l;[W_0^{1,2}(S)]')$ for $\epsilon_n \to 0$ and $q \in [0,1)$, which leads to convergences $C_3(\epsilon_n) \to 0$,

$$\frac{1}{\epsilon_n^q}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{g}_{1,\epsilon_n}) \to 0,\ \frac{1}{\epsilon_n^q}(\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{g}_{1,\epsilon_n}) \to 0$$

in $L^2(\Omega)$ for $q \in [0,1)$,

$$\frac{1}{\epsilon_n^{q_1}}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}) \to 0,\ \frac{1}{\epsilon_n^{q_1}}(\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}) \to 0$$

in $L^2(\Omega)$ for $q_1 \in [0,2)$ and $\epsilon_n \to 0$, and subsequently to convergences of $C_4(\epsilon_n) \to 0$ and $C_6(\epsilon_n) \to 0$,

$$\frac{1}{\epsilon_n^{q_2}}(\partial_j \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n}) \to 0 \text{ in } L^2(\Omega),\ j = 2,3,$$

for $q_2 \in [0, 1-r)$, $r \in \left(0, \frac{1}{3}\right)$ and $\epsilon_n \to 0$, and to $C_5(\epsilon_n) \to 0$ because of $4r - 1 < 1 - r$ for $r \in \left(0, \frac{1}{3}\right)$. $C_2(\epsilon_n) \to 0$ is a consequence of (5.1.40). Hence, we can conclude that

$$\frac{1}{\epsilon_n^{2r}}\Big(\|\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}\|_2 + \|\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}\|_2\Big) \to 0 \tag{5.1.54}$$

for $r \in \left(0, \frac{1}{3}\right)$, which, together with (5.1.49), implies (5.1.46) and thus (5.1.42). Now, it remains to prove (5.1.45). Since

$$\partial_1 \mathbf{u}_{*,\epsilon_n} = (\partial_1 \mathbf{u}_{*,\epsilon_n} \cdot \mathbf{t}_{\epsilon_n})\mathbf{t}_{\epsilon_n} + (\partial_1 \mathbf{u}_{*,\epsilon_n} \cdot \mathbf{n}_{\epsilon_n})\mathbf{n}_{\epsilon_n} + (\partial_1 \mathbf{u}_{*,\epsilon_n} \cdot \mathbf{b}_{\epsilon_n})\mathbf{b}_{\epsilon_n},$$

it is enough to show that

$$\begin{aligned}
(\partial_1 \mathbf{u}_{*,\epsilon_n} \cdot \mathbf{t}_{\epsilon_n})\mathbf{t}_{\epsilon_n} &\rightharpoonup (\partial_3 \zeta_{12} - \partial_2 \zeta_{13})\mathbf{t} \text{ in } L^2(0,l;[W_0^{1,2}(S)^3]'),\\
(\partial_1 \mathbf{u}_{*,\epsilon_n} \cdot \mathbf{n}_{\epsilon_n})\mathbf{n}_{\epsilon_n} &\rightharpoonup -\partial_3 \zeta_{11}\mathbf{n} \text{ in } L^2(0,l;[W_0^{1,2}(S)^3]'),\\
(\partial_1 \mathbf{u}_{*,\epsilon_n} \cdot \mathbf{b}_{\epsilon_n})\mathbf{b}_{\epsilon_n} &\rightharpoonup \partial_2 \zeta_{11}\mathbf{n} \text{ in } L^2(0,l;[W_0^{1,2}(S)^3]')
\end{aligned}$$

for $\epsilon_n \to 0$. We check only the first convergence. The next ones can be proved in almost the same way. Since $\mathbf{t}$ is a bounded function depending only on x_1, then (5.1.42) yields

$$(\partial_1 \mathbf{u}_{*,\epsilon_n} \cdot \mathbf{t}_{\epsilon_n})\mathbf{t} \rightharpoonup (\partial_3 \zeta_{12} - \partial_2 \zeta_{13})\mathbf{t} \text{ in } L^2(0,l;[W_0^{1,2}(S)^3]').$$

It remains to show that

$$(\partial_1 \mathbf{u}_{*,\epsilon_n} \cdot \mathbf{t}_{\epsilon_n})\mathbf{t}_{\epsilon_n} - (\partial_1 \mathbf{u}_{*,\epsilon_n} \cdot \mathbf{t}_{\epsilon_n})\mathbf{t} \rightharpoonup 0 \text{ in } L^2(0,l;[W_0^{1,2}(S)^3]')$$

for $\epsilon_n \to 0$, which follows from the estimate

$$\left|\int_\Omega (\partial_1 \mathbf{u}_{*,\epsilon_n} \cdot \mathbf{t}_{\epsilon_n})(\mathbf{t}_{\epsilon_n} - \mathbf{t})\varphi \, dx\right| \le$$

$$\le C\left(\int_0^l |\mathbf{t}_{\epsilon_n}(x_1) - \mathbf{t}(x_1)|^2 \|\varphi(x_1)\|_{1,2,S}^2 \, dx_1\right)^{\frac{1}{2}} \to 0,$$

for $\epsilon_n \to 0$ and for arbitrary but fixed function $\varphi \in L^2(0,l;W_0^{1,2}(S))$, because $|\mathbf{t}_{\epsilon_n}| = |\mathbf{t}| = 1$, $\forall \epsilon_n \in (0,1)$, $\mathbf{t}_{\epsilon_n} \to \mathbf{t}$ pointwisely in $[0,l] \setminus D$ and thus we can use the Lebesgue theorem.

5.
$$\partial_j \mathbf{u}_{*,\epsilon_n} \rightharpoonup \mathbf{0} \ in \ L^2(0,l;[W_0^{1,2}(S)^3]'), \ j = 2,3, \tag{5.1.55}$$

and $\mathbf{u}_{*,\epsilon_n}|_{x_1=0} = \mathbf{u}_{*,\epsilon_n}|_{x_1=l} = 0$ *in the sense of the space* $C([0,l];[W_0^{1,2}(S)^3]')$.

Since $\phi_{\epsilon_n}|_{x_1=0} = \phi_{\epsilon_n}|_{x_1=l} = 0$ for all $\epsilon_n \in (0,1)$ in the sense of the space $C([0,l];[W_0^{1,2}(S)^3]')$ (see 3.), $\mathbf{u}_{\epsilon_n} \in V(\Omega)$ and since the functions $\mathbf{g}_{1,\epsilon_n}$, $\mathbf{t}_{\epsilon_n}$, $\mathbf{n}_{\epsilon_n}$, $\mathbf{b}_{\epsilon_n}$ belong to $C^\infty(\overline{\Omega})^3$, we can use the definition (5.1.41) of the functions $\mathbf{u}_{*,\epsilon_n}$ and applying Proposition 5.1.3, we get that $\mathbf{u}_{*,\epsilon_n}|_{x_1=0} = \mathbf{u}_{*,\epsilon_n}|_{x_1=l} = 0$ in the sense of the space $C([0,l];[W_0^{1,2}(S)^3]')$.

It remains to show (5.1.55). Using the definition (5.1.41) of the functions $\mathbf{u}_{*,\epsilon_n}$, we obtain the identity

$$\begin{aligned}\partial_j \mathbf{u}_{*,\epsilon_n} = -\partial_j \phi_{\epsilon_n} \mathbf{t}_{\epsilon_n} + \partial_j(\partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n})\mathbf{n}_{\epsilon_n} - \partial_j(\partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n})\mathbf{b}_{\epsilon_n} - \\ -\partial_j \left(\frac{1}{\epsilon_n}(\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{g}_{1,\epsilon_n}) + \partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n} \right) \mathbf{n}_{\epsilon_n} + \\ +\partial_j \left(\frac{1}{\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{g}_{1,\epsilon_n}) + \partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n} \right) \mathbf{b}_{\epsilon_n}\end{aligned} \tag{5.1.56}$$

in $L^2(0,l;[W_0^{1,2}(S)^3]')$, $j = 2,3$. From (5.1.28), (5.1.5)–(5.1.8), (5.1.40), (4.3.3) and from the fact that the functions $\mathbf{t}_{\epsilon_n}$, $\mathbf{n}_{\epsilon_n}$, $\mathbf{b}_{\epsilon_n}$ are bounded in $L^\infty(0,l)^3$, it follows that

$$\partial_j \phi_{\epsilon_n} \mathbf{t}_{\epsilon_n} \to \mathbf{0} \text{ in } L^2(0,l;[W_0^{1,2}(S)^3]'), \ j = 2,3,$$

$$\partial_j \left(\frac{1}{\epsilon_n}(\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{g}_{1,\epsilon_n}) + \partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n} \right) \mathbf{n}_{\epsilon_n} \to \mathbf{0} \text{ in } L^2(0,l;[W_0^{1,2}(S)^3]'),$$

$$\partial_j \left(\frac{1}{\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{g}_{1,\epsilon_n}) + \partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n} \right) \mathbf{b}_{\epsilon_n} \to \mathbf{0} \text{ in } L^2(0,l;[W_0^{1,2}(S)^3]')$$

for $j = 2,3$ and $\epsilon_n \to 0$. We can see from (5.1.56) that it remains to prove that

$$\begin{aligned}\partial_j(\partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n})\mathbf{n}_{\epsilon_n} &\rightharpoonup \mathbf{0} \text{ in } L^2(0,l;[W_0^{1,2}(S)^3]'), \ j = 2,3, \\ \partial_j(\partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n})\mathbf{b}_{\epsilon_n} &\rightharpoonup \mathbf{0} \text{ in } L^2(0,l;[W_0^{1,2}(S)^3]'), \ j = 2,3\end{aligned}$$

for $\epsilon_n \to 0$. From (5.1.27), it follows that $\partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n} \rightharpoonup \partial_1 \mathbf{u} \cdot \mathbf{n}$ in $L^2(\Omega)$ because $\mathbf{n}$ is a bounded function. Similarly as above we can deduce

$$\partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n} \rightharpoonup \partial_1 \mathbf{u} \cdot \mathbf{n} \text{ and } (\partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n})\mathbf{b}_{\epsilon_n} \rightharpoonup (\partial_1 \mathbf{u} \cdot \mathbf{n})\mathbf{b} \text{ in } L^2(\Omega)^3.$$

The second convergence can be proved in the same way. In point 1., we proved that the function $\mathbf{u}$ depends only on x_1 and hence

$$\partial_j(\partial_1 \mathbf{u} \cdot \mathbf{n})\mathbf{b} = \mathbf{0}, \ \partial_j(\partial_1 \mathbf{u} \cdot \mathbf{b})\mathbf{n} = \mathbf{0}, \ j = 2,3.$$

6.

$$\partial_i \mathbf{u}_{*,\epsilon_n} \rightharpoonup \partial_i \mathbf{u}_* \ in \ L^2(0,l;[W_0^{1,2}(S)^3]'), \ i = 1,2,3, \tag{5.1.57}$$
$$\mathbf{u}_{*,\epsilon_n} \rightharpoonup \mathbf{u}_* \ in \ L^2(\Omega)^3, \tag{5.1.58}$$
$$\mathbf{u}_{*,\epsilon_n} \to \mathbf{u}_* \ in \ C_0([0,l];[W_0^{1,2}(S)^3]') \tag{5.1.59}$$

for $\epsilon_n \to 0$*, and* $\mathbf{u}_* \in W_0^{1,2}(0,l)^3$*, where*

$$\mathbf{u}_*(x_1) = \int_0^{x_1} [(\partial_3\zeta_{12}(z_1,x_2,x_3) - \partial_2\zeta_{13}(z_1,x_2,x_3))\mathbf{t}(z_1) -$$

$$-\partial_3\zeta_{11}(z_1,x_2,x_3)\mathbf{n}(z_1) + \partial_2\zeta_{11}(z_1,x_2,x_3)\mathbf{b}(z_1)]\ dz_1 \qquad (5.1.60)$$

for $(x_1,x_2,x_3) \in (0,l)\times S$*. In addition,*

$$\phi_{\epsilon_n} \rightharpoonup \phi = \mathbf{u}_* \cdot \mathbf{t} \ \ in\ L^2(\Omega) \qquad (5.1.61)$$

for $\epsilon_n \to 0$ *and* ϕ *is piecewise continuous.*

Points 4. and 5. enable us to use Proposition 5.1.3 and 5.1.4 to prove (5.1.57)–(5.1.60) and $\mathbf{u}_* \in W_0^{1,2}(0,l)^3$. From (5.1.41), it follows that $\phi_{\epsilon_n} = -\mathbf{u}_{*,\epsilon_n}\cdot\mathbf{t}_{\epsilon_n}$. Then (5.1.61) easily follows from (5.1.58) using the pointwise convergence on $[0,l]\setminus D$ of the functions $\mathbf{t}_{\epsilon_n}$.

7. *Function* $\mathbf{u}$ *determined by* (5.1.27) *and function* ϕ *by* (5.1.61) *form a couple such that* $\langle \mathbf{u},\phi\rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$*. In addition* $\mathbf{u}_*$ *satisfies* (5.1.31)–(5.1.33).

   To prove that $\langle \mathbf{u},\phi\rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$, it is enough to check that $\mathbf{u} = \widehat{\mathbf{u}}$, where

$$\widehat{\mathbf{u}}(x_1) = \int_0^{x_1} [-(\mathbf{u}_*\cdot\mathbf{b})\mathbf{n} + (\mathbf{u}_*\cdot\mathbf{n})\mathbf{b}]\ dz_1,\ \ x_1\in[0,l]$$

(see (4.2.17) and Proposition 4.4.1). We define the function $\widehat{\mathbf{u}}_{\epsilon_n}$ by

$$\widehat{\mathbf{u}}_{\epsilon_n}(x_1,x_2,x_3) := \int_0^{x_1} [-(\mathbf{u}_{*,\epsilon_n}(z_1,x_2,x_3)\cdot\mathbf{b}_{\epsilon_n}(z_1))\mathbf{n}_{\epsilon_n}(z_1) +$$

$$+(\mathbf{u}_{*,\epsilon_n}(z_1,x_2,x_3)\cdot\mathbf{n}_{\epsilon_n}(z_1))\mathbf{b}_{\epsilon_n}(z_1)]\ dz_1,\ \ (x_1,x_2,x_3)\in[0,l]\times S. \qquad (5.1.62)$$

The definition (5.1.41) of the function $\mathbf{u}_{*,\epsilon_n}$ together with (5.1.62) enable us to express the function $\widehat{\mathbf{u}}_{\epsilon_n}$ by

$$\widehat{\mathbf{u}}_{\epsilon_n} = -\int_0^{x_1}\left[\frac{1}{\epsilon_n}(\partial_2\mathbf{u}_{\epsilon_n}\cdot\mathbf{g}_{1,\epsilon_n})\mathbf{n}_{\epsilon_n} + \frac{1}{\epsilon_n}(\partial_3\mathbf{u}_{\epsilon_n}\cdot\mathbf{g}_{1,\epsilon_n})\mathbf{b}_{\epsilon_n}\right]\ dz_1, \qquad (5.1.63)$$

where we omit to write the variables (z_1,x_2,x_3) and (z_1) on the right-hand side to simplify the notation. Using (5.1.63), we can deduce that

$$\begin{aligned}
\mathbf{u}_{\epsilon_n} &= \int_0^{x_1}\partial_1\mathbf{u}_{\epsilon_n}\ dz_1 = \\
&= \int_0^{x_1}(\partial_1\mathbf{u}_{\epsilon_n}\cdot\mathbf{t}_{\epsilon_n})\mathbf{t}_{\epsilon_n} + (\partial_1\mathbf{u}_{\epsilon_n}\cdot\mathbf{n}_{\epsilon_n})\mathbf{n}_{\epsilon_n} + (\partial_1\mathbf{u}_{\epsilon_n}\cdot\mathbf{b}_{\epsilon_n})\mathbf{b}_{\epsilon_n}\ dz_1 = \\
&= \widehat{\mathbf{u}}_{\epsilon_n} + \int_0^{x_1}\left[(\partial_1\mathbf{u}_{\epsilon_n}\cdot\mathbf{t}_{\epsilon_n})\mathbf{t}_{\epsilon_n} + \left(\frac{\partial_2\mathbf{u}_{\epsilon_n}\cdot\mathbf{g}_{1,\epsilon_n}}{\epsilon_n} + \partial_1\mathbf{u}_{\epsilon_n}\cdot\mathbf{n}_{\epsilon_n}\right)\mathbf{n}_{\epsilon_n} + \right. \\
&+ \left.\left(\frac{\partial_3\mathbf{u}_{\epsilon_n}\cdot\mathbf{g}_{1,\epsilon_n}}{\epsilon_n} + \partial_1\mathbf{u}_{\epsilon_n}\cdot\mathbf{b}_{\epsilon_n}\right)\mathbf{b}_{\epsilon_n}\right]\ dz_1.
\end{aligned}$$

Hence and from (5.1.28), (5.1.5)–(5.1.8) we get (similarly as in 4.)

$$\partial_1\widehat{\mathbf{u}}_{\epsilon_n} - \partial_1\mathbf{u}_{\epsilon_n} \to 0 \text{ in } L^2(\Omega)^3 \text{ and } \widehat{\mathbf{u}}_{\epsilon_n} - \mathbf{u}_{\epsilon_n} \to 0 \text{ in } C([0,l];L^2(S)^3)$$

for $\epsilon_n \to 0$. Since, in addition, $\mathbf{u}_{\epsilon_n} \rightharpoonup \mathbf{u}$ in $W^{1,2}(\Omega)^3$ and $\mathbf{u}\in W_0^{1,2}(0,l)^3$, we can conclude that $\mathbf{u} = \widehat{\mathbf{u}}$ a.e. in $[0,l]$ and thus

$$\mathbf{u}(x_1) = \int_0^{x_1}[-(\mathbf{u}_*\cdot\mathbf{b})\mathbf{n} + (\mathbf{u}_*\cdot\mathbf{n})\mathbf{b}]\ dz_1,\ \ x_1\in[0,l],$$

and

$$\mathbf{u}(l) = \int_0^l [-(\mathbf{u}_* \cdot \mathbf{b})\mathbf{n} + (\mathbf{u}_* \cdot \mathbf{n})\mathbf{b}]\, dx_1 = 0.$$

Hence, from (4.2.17) and Proposition 4.4.1, we get that $\langle \mathbf{u}, \phi \rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$. The rest follows from (5.1.60).

8. *Let* $\frac{1}{\epsilon_n}\omega^{\epsilon_n}(\mathbf{u}_{\epsilon_n}) \to \zeta$ *in* $L^2(\Omega)^9$. *Then*

$$\mathbf{u}_{\epsilon_n} \to \mathbf{u} \textit{ in } W^{1,2}(\Omega)^3 \tag{5.1.64}$$

for $\epsilon_n \to 0$.

From (5.1.5)–(5.1.8) it follows

$$\partial_j \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n} \to 0,\ j = 2,3,\ \partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n} \to 0 \text{ and } \partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n} \to 0$$

in $L^2(\Omega)$. To prove that $\partial_2 \mathbf{u}_{\epsilon_n}$ and $\partial_3 \mathbf{u}_{\epsilon_n}$ converge strongly in $L^2(\Omega)^3$, we must verify the strong convergence of the functions $\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n}$ and $\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}$ to zero in $L^2(\Omega)$, which follows from (5.1.54). The rest of the proof is a consequence of the identity

$$\partial_j \mathbf{u}_{\epsilon_n} = (\partial_j \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n})\mathbf{t}_{\epsilon_n} + (\partial_j \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n})\mathbf{n}_{\epsilon_n} + (\partial_j \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n})\mathbf{b}_{\epsilon_n},\ j = 2,3,$$

because $|\mathbf{t}_{\epsilon_n}| = |\mathbf{n}_{\epsilon_n}| = |\mathbf{b}_{\epsilon_n}| = 1$ for all $n \in \mathbb{N}$.

It remains to investigate the functions $\partial_1 \mathbf{u}_{\epsilon_n}$. We know from (4.1.5)–(5.1.8) that

$$\partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n} \to 0 \text{ in } L^2(\Omega).$$

Similarly as for $\partial_j \mathbf{u}_{\epsilon_n}$ it is enough to prove the strong convergences

$$(\partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n})\mathbf{n}_{\epsilon_n} \to (\partial_1 \mathbf{u} \cdot \mathbf{n})\mathbf{n} \text{ and } (\partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n})\mathbf{b}_{\epsilon_n} \to (\partial_1 \mathbf{u} \cdot \mathbf{b})\mathbf{b}$$

in $L^2(\Omega)^3$. Since $\mathbf{n}_{\epsilon_n} \to \mathbf{n}$ pointwisely in $[0,l] \setminus D$ and $|\mathbf{n}_{\epsilon_n}| = 1$ it is enough to check that

$$\partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n} \to \partial_1 \mathbf{u} \cdot \mathbf{n},\ \partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n} \to \partial_1 \mathbf{u} \cdot \mathbf{b} \text{ in } L^2(\Omega). \tag{5.1.65}$$

Due to (5.1.5)–(5.1.8) the problem is equivalent to the convergences

$$\begin{aligned} \frac{1}{\epsilon_n}(\partial_2 \mathbf{u}_{\epsilon_n} \cdot \mathbf{g}_{1,\epsilon_n}) &\to -\partial_1 \mathbf{u} \cdot \mathbf{n} \text{ in } L^2(\Omega), \\ \frac{1}{\epsilon_n}(\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{g}_{1,\epsilon_n}) &\to -\partial_1 \mathbf{u} \cdot \mathbf{b} \text{ in } L^2(\Omega). \end{aligned}$$

The convergences follow from the convergences of the functions $\mathbf{u}_{*,\epsilon_n}$

$$\begin{aligned} \mathbf{u}_{*,\epsilon_n} \cdot \mathbf{b}_{\epsilon_n} &\to \mathbf{u}_* \cdot \mathbf{b} \text{ in } L^2(\Omega), \\ \mathbf{u}_{*,\epsilon_n} \cdot \mathbf{n}_{\epsilon_n} &\to \mathbf{u}_* \cdot \mathbf{n} \text{ in } L^2(\Omega) \end{aligned}$$

that follow from (5.1.41) and the fact that $\langle \mathbf{u}, \phi \rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ (see (4.2.17)). Due to the pointwise convergences of sequences $\{\mathbf{b}_{\epsilon_n}\}_{n=1}^\infty$ and $\{\mathbf{n}_{\epsilon_n}\}_{n=1}^\infty$ and their boundedness we can restrict ourselves to the proof of

$$\mathbf{u}_{*,\epsilon_n} \to \mathbf{u}_* \text{ in } L^2(\Omega)^3.$$

To check the convergence, we use the inequality (C is independent of v)

$$\|v\|_2 \le C(\|v\|_{[W_0^{1,2}(\Omega)]'} + \|\nabla v\|_{[W_0^{1,2}(\Omega)^3]'}),\ \forall v \in L^2(\Omega)$$

(see [14, p. 189]). First, it follows from (5.1.58) that

$$\mathbf{u}_{*,\epsilon_n} \to \mathbf{u}_* \text{ in } [W_0^{1,2}(\Omega)^3]'.$$

In the second step, we show that

$$\nabla \mathbf{u}_{*,\epsilon_n} \to \nabla \mathbf{u}_* \text{ in } [W_0^{1,2}(\Omega)^9]'.$$

Since we suppose that $\frac{1}{\epsilon_n}\omega^{\epsilon_n}(\mathbf{u}_{\epsilon_n}) \to \zeta$ in $L^2(\Omega)^9$ and thus $\partial_j \frac{1}{\epsilon_n}\omega^{\epsilon_n}(\mathbf{u}_{\epsilon_n}) \to \partial_j \zeta$ for $\epsilon_n \to 0$ in the space $L^2(0,l;[W_0^{1,2}(S)^9]')$, $j = 2, 3$, we can use (5.1.5)–(5.1.8) together with (5.1.46)–(5.1.48) and (5.1.60) to deduce

$$\begin{aligned}
\partial_1 \mathbf{u}_{*,\epsilon_n} \cdot \mathbf{t}_{\epsilon_n} &\to \partial_1 \mathbf{u}_* \cdot \mathbf{t} \text{ in } L^2(0,l;[W_0^{1,2}(S)]'),\\
\partial_1 \mathbf{u}_{*,\epsilon_n} \cdot \mathbf{n}_{\epsilon_n} &\to \partial_1 \mathbf{u}_* \cdot \mathbf{n} \text{ in } L^2(0,l;[W_0^{1,2}(S)]'),\\
\partial_1 \mathbf{u}_{*,\epsilon_n} \cdot \mathbf{b}_{\epsilon_n} &\to \partial_1 \mathbf{u}_* \cdot \mathbf{b} \text{ in } L^2(0,l;[W_0^{1,2}(S)]').
\end{aligned}$$

Using again the properties of $\mathbf{t}_{\epsilon_n}$, $\mathbf{n}_{\epsilon_n}$ and $\mathbf{b}_{\epsilon_n}$ we arrive at

$$\partial_1 \mathbf{u}_{*,\epsilon_n} \to \partial_1 \mathbf{u}_* \text{ in } L^2(0,l;[W_0^{1,2}(S)^3]')$$

and thus strongly in $[W_0^{1,2}(\Omega)^3]'$.
Further, we want to show that

$$\partial_j \mathbf{u}_{*,\epsilon_n} \to 0 \text{ in } [W_0^{1,2}(\Omega)^3]',\ j = 2, 3,$$

for $\epsilon_n \to 0$. If we take (5.1.40), (5.1.56) and its subsequent convergences we can restrict ourselves to the proof of

$$\begin{aligned}
\partial_j(\partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n})\mathbf{n}_{\epsilon_n} &\to 0 \text{ in } [W_0^{1,2}(\Omega)^3]',\ j = 2, 3,\\
\partial_j(\partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n})\mathbf{b}_{\epsilon_n} &\to 0 \text{ in } [W_0^{1,2}(\Omega)^3]',\ j = 2, 3,
\end{aligned}$$

for $\epsilon_n \to 0$. The relations in (4.2.3) provide us with

$$\partial_j(\partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n})\mathbf{b}_{\epsilon_n} = \partial_j \partial_1(\mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n})\mathbf{b}_{\epsilon_n} - \partial_j(\mathbf{u}_{\epsilon_n} \cdot \mathbf{n}'_{\epsilon_n})\mathbf{b}_{\epsilon_n} =$$

$$= \partial_1(\partial_j \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n})\mathbf{b}_{\epsilon_n} + \beta_{\epsilon_n}(\partial_j \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n})\mathbf{b}_{\epsilon_n} + \gamma_{\epsilon_n}(\partial_j \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n})\mathbf{b}_{\epsilon_n} \quad (5.1.66)$$

for $j = 2, 3$. An analogous relation can be derived for $\partial_j(\partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n})\mathbf{n}_{\epsilon_n}$. We pay attention only to (5.1.66) because the proof for the second function is similar. In view of (4.3.6), (5.1.5)–(5.1.8) and (5.1.54), we can deduce

$$\beta_{\epsilon_n}(\partial_j \mathbf{u}_{\epsilon_n} \cdot \mathbf{t}_{\epsilon_n})\mathbf{b}_{\epsilon_n} \to 0,\ \gamma_{\epsilon_n}(\partial_j \mathbf{u}_{\epsilon_n} \cdot \mathbf{b}_{\epsilon_n})\mathbf{b}_{\epsilon_n} \to 0 \text{ in } L^2(\Omega)^3.$$

At the end, we prove that

$$\partial_1(\partial_j \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n})\mathbf{b}_{\epsilon_n} \to 0 \text{ in } [W_0^{1,2}(\Omega)^3]',\ j = 2, 3.$$

It immediately follows from the estimate (using (4.2.3) and (4.3.6))

$$\left| \int_\Omega (\partial_j \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}) \partial_1(\mathbf{b}_{\epsilon_n} \varphi)\, dx \right| \leq \frac{C}{\epsilon_n^r} \|\partial_j \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}\|_2 \|\varphi\|_2 + \|\partial_j \mathbf{u}_{\epsilon_n} \cdot \mathbf{n}_{\epsilon_n}\|_2 \|\partial_1 \varphi\|_2$$

for $\epsilon_n \to 0$, $r \in \left(0, \frac{1}{3}\right)$, $j = 2, 3$, and any function $\varphi \in W_0^{1,2}(\Omega)$. Hence we have

$$\mathbf{u}_{*,\epsilon_n} \to \mathbf{u}_* \text{ in } L^2(\Omega)^3.$$

□

The following theorem is a version of Korn's inequality that is necessary for the derivation of a priori estimates in the next sections.

Theorem 5.1.7 *There exists a constant $C > 0$ independent of ϵ such that*

$$\|\mathbf{v}\|_{1,2} \le \frac{C}{\epsilon}\|\omega^\epsilon(\mathbf{v})\|_2,\ \forall \mathbf{v} \in V(\Omega) \text{ and } \forall \epsilon \in (0,1). \tag{5.1.67}$$

Proof: Let us assume the contrary, i.e., there exist the sequences $\{\epsilon_n\}_{n=1}^\infty$, $\epsilon_n \in (0, 1/n)$, and $\{\mathbf{v}_{\epsilon_n}\}_{n=1}^\infty$, $\|\mathbf{v}_{\epsilon_n}\|_{1,2} = 1$, such that

$$\frac{1}{\epsilon_n}\|\omega^{\epsilon_n}(\mathbf{v}_{\epsilon_n})\|_2 \le \frac{1}{n}.$$

Hence (passing to a subsequence if it is necessary),

$$\mathbf{v}_{\epsilon_n} \rightharpoonup \mathbf{v} \text{ in } W^{1,2}(\Omega)^3 \text{ and } \frac{1}{\epsilon_n}\omega^{\epsilon_n}(\mathbf{v}_{\epsilon_n}) \to 0 \text{ in } L^2(\Omega)^9.$$

Due to Proposition 5.1.6 we arrive at

$$\mathbf{v}_{\epsilon_n} \to \mathbf{v} \text{ in } W^{1,2}(\Omega)^3,\ \frac{1}{2\epsilon_n}(\partial_2\mathbf{v}_{\epsilon_n}\cdot\mathbf{b}_{\epsilon_n} - \partial_3\mathbf{v}_{\epsilon_n}\cdot\mathbf{n}_{\epsilon_n}) \to \psi \text{ in } L^2(\Omega)$$

and thus

$$\mathbf{v}'\cdot\mathbf{t} = 0,\ \mathbf{v}_*'\cdot\mathbf{t} = 0,\ \mathbf{v}_*'\cdot\mathbf{n} = 0,\ \mathbf{v}_*'\cdot\mathbf{b} = 0,\ \psi = 0. \tag{5.1.68}$$

From the same proposition and the definition (4.2.17) of the space $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$, it follows that $\langle\mathbf{v},\psi\rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ and thus $\mathbf{v} \in W_0^{1,2}(0,l)^3$ and $\mathbf{v}_* \in W_0^{1,2}(0,l)^3$. In view of (5.1.68), we have that $\mathbf{v}_* = \mathbf{0}$ and thus $\mathbf{v} = \mathbf{0}$, a contradiction. □

At the end of the section, we pay attention to the limit tensor ζ from (5.1.28) if $\mathbf{u}_{\epsilon_n}$ are solutions to respective elasticity problems. Its properties enable us to derive limit equations. To find the form of the tensor ζ, we must obtain the corresponding equations for its components. Let us start with the static linear elasticity.

Proposition 5.1.8 *Let the tensor ζ be the limit determined by* (5.1.28) *and $\mathbf{u}_{\epsilon_n}$ be solutions to the equations*

$$\int_\Omega A_{\epsilon_n}^{ijkl}\frac{1}{\epsilon_n}\omega_{kl}^{\epsilon_n}(\mathbf{u}_{\epsilon_n})\epsilon_n\omega_{ij}^{\epsilon_n}(\mathbf{v})\,d_{\epsilon_n}\ dx = \epsilon_n^2\int_\Omega \mathbf{f}\cdot\mathbf{v}\,d_{\epsilon_n}\ dx+$$

$$+\epsilon_n^2\int_0^l\int_{\partial S}\mathbf{h}\cdot\mathbf{v}\,d_{\epsilon_n}\epsilon_n\sqrt{\nu_j o^{ij,\epsilon_n}\nu_j}\ dSdx_1,\ \forall\mathbf{v}\in V(\Omega), \tag{5.1.69}$$

(*see* [199] *for more details about the problem*). *Then ζ satisfies the equation*

$$\int_\Omega A_0^{ijkl}\zeta_{kl}\theta_{ij}^0(\mathbf{v})\ dx = 0,\ \forall\mathbf{v}\in L^2(0,l;W^{1,2}(S)^3), \tag{5.1.70}$$

where the tensor $\theta^0(\mathbf{v})$ is defined by

$$\theta^0(\mathbf{v}) = \begin{pmatrix} 0 & \frac{\partial_2\mathbf{v}\cdot\mathbf{t}}{2} & \frac{\partial_3\mathbf{v}\cdot\mathbf{t}}{2} \\ \frac{\partial_2\mathbf{v}\cdot\mathbf{t}}{2} & \partial_2\mathbf{v}\cdot\mathbf{n} & \frac{\partial_2\mathbf{v}\cdot\mathbf{b}+\partial_3\mathbf{v}\cdot\mathbf{n}}{2} \\ \frac{\partial_3\mathbf{v}\cdot\mathbf{t}}{2} & \frac{\partial_2\mathbf{v}\cdot\mathbf{b}+\partial_3\mathbf{v}\cdot\mathbf{n}}{2} & \partial_3\mathbf{v}\cdot\mathbf{b}\end{pmatrix}. \tag{5.1.71}$$

Proof: In the proof, we will use ϵ instead of ϵ_n to simplify the notation. We want to pass from the equations (5.1.69) to the equation

$$\int_\Omega A_0^{ijkl}\zeta_{kl}\theta_{ij}^0(\mathbf{v})\ dx = 0,\ \forall\mathbf{v}\in V(\Omega), \tag{5.1.72}$$

where the tensor $\theta^0(\mathbf{v})$ is defined by (5.1.71). We show that the tensor $\theta^0(\mathbf{v})$ is the limit state of the tensors $\theta^\epsilon(\mathbf{v}) + \epsilon\kappa^\epsilon(\mathbf{v})$ for $\epsilon \to 0$ (see (5.1.5)–(5.1.8)). Since the functions $\mathbf{g}_{1,\epsilon}$, $\mathbf{n}_\epsilon$ and $\mathbf{b}_\epsilon$ are bounded in $L^\infty(\Omega)^3$ or $L^\infty(0,l)^3$, it is easily seen that $\epsilon\kappa^\epsilon(\mathbf{v}) \to 0$ in $L^2(\Omega)^9$ (see (5.1.8)). Thus it remains to show that $\theta^\epsilon(\mathbf{v}) \to \theta^0(\mathbf{v})$ in $L^2(\Omega)^9$ for $\epsilon \to 0$. Since we know that $\mathbf{g}_{1,\epsilon} \to \mathbf{t}$, $\mathbf{n}_\epsilon \to \mathbf{n}$, and $\mathbf{b}_\epsilon \to \mathbf{b}$ pointwisely in $\Omega \setminus (S \times D)$ or in $[0,l] \setminus D$, respectively, and are bounded in $L^\infty(\Omega)^3$ or $L^\infty(0,l)^3$, respectively, we can use (5.1.6)–(5.1.7) to prove the above-mentioned strong convergence.

Using the definition of the tensors $(A_\epsilon^{ijkl})_{i,j,k,l=1}^3$ (see (5.1.9)), we can easily check by (4.2.6)–(4.2.12) that

$$A_\epsilon^{ijkl} \to A_0^{ijkl} \text{ in } C(\overline{\Omega}), \text{ where } A_0^{ijkl} = \lambda\delta^{ij}\delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}) \qquad (5.1.73)$$

for $i,j,k,l = 1,2,3$. The rest of the proof follows from density of the space $V(\Omega)$ in $L^2(0,l;W^{1,2}(S)^3)$ and from (5.1.71) and (5.1.72). $\square$

Before the following corollary, we introduce the notation:

$$\zeta_{22}^H := \zeta_{22} + \frac{1}{2}\frac{\lambda}{\lambda+\mu}\zeta_{11},\ \zeta_{33}^H := \zeta_{33} + \frac{1}{2}\frac{\lambda}{\lambda+\mu}\zeta_{11},\ \zeta_{23}^H := \zeta_{23}. \qquad (5.1.74)$$

Corollary 5.1.9 *We have*

$$\int_S \zeta_{12} = \int_S \zeta_{13} = \int_S \zeta_{12}x_2 = \int_S \zeta_{13}x_3 = \int_S [\zeta_{12}x_3 + \zeta_{13}x_2] = 0, \qquad (5.1.75)$$

$$\int_S \zeta_{23}^H = \int_S \zeta_{23}^H x_2 = \int_S \zeta_{23}^H x_3 = 0 \qquad (5.1.76)$$

and

$$\int_S [\zeta_{22}^H + \zeta_{33}^H] = \int_S [\zeta_{22}^H + \zeta_{33}^H]x_2 = \int_S [\zeta_{22}^H + \zeta_{33}^H]x_3 = 0. \qquad (5.1.77)$$

Proof: Let $v \in L^2(0,l)$ be an arbitrary but fixed function and $\mathbf{v} = v\mathbf{t}$. Testing equation (5.1.70) with functions $\mathbf{v}x_2$, $\mathbf{v}x_3$, $\mathbf{v}x_2^2/2$, $\mathbf{v}x_3^2/2$ and $\mathbf{v}x_2x_3$ we can derive (5.1.75).

Let us take now some arbitrary function $\mathbf{v} \in L^2(0,l;W^{1,2}(S)^3)$ such that $\mathbf{v}\cdot\mathbf{t} = \mathbf{v}\cdot\mathbf{b} = 0$. Then we can derive from (5.1.70) and (5.1.71) that

$$\int_\Omega [(\lambda(\zeta_{11} + \zeta_{22} + \zeta_{33}) + 2\mu\zeta_{22})(\partial_2\mathbf{v}\cdot\mathbf{n}) + 2\mu\zeta_{23}(\partial_3\mathbf{v}\cdot\mathbf{n})]\, dx = 0. \qquad (5.1.78)$$

Analogously we deduce for arbitrary function $\mathbf{v} \in L^2(0,l;W^{1,2}(S)^3)$ that satisfies $\mathbf{v}\cdot\mathbf{t} = \mathbf{v}\cdot\mathbf{n} = 0$ that

$$\int_\Omega [(\lambda(\zeta_{11} + \zeta_{22} + \zeta_{33}) + 2\mu\zeta_{33})(\partial_3\mathbf{v}\cdot\mathbf{b}) + 2\mu\zeta_{23}(\partial_2\mathbf{v}\cdot\mathbf{b})]\, dx = 0. \qquad (5.1.79)$$

Using notation (5.1.74) we can transform (5.1.78) and (5.1.79) as

$$\int_\Omega [(\lambda(\zeta_{22}^H + \zeta_{33}^H) + 2\mu\zeta_{22}^H)(\partial_2\mathbf{v}\cdot\mathbf{n}) + 2\mu\zeta_{23}^H(\partial_3\mathbf{v}\cdot\mathbf{n})]\, dx = 0 \qquad (5.1.80)$$

and

$$\int_\Omega [(\lambda(\zeta_{22}^H + \zeta_{33}^H) + 2\mu\zeta_{33}^H)(\partial_3\mathbf{v}\cdot\mathbf{b}) + 2\mu\zeta_{23}^H(\partial_2\mathbf{v}\cdot\mathbf{b})]\, dx = 0, \qquad (5.1.81)$$

respectively. Taking $\mathbf{v}x_3$, $\mathbf{v}x_3^2/2$ and $\mathbf{v}x_2^2/2$, where $\mathbf{v} = v\mathbf{n}$ or $\mathbf{v} = v\mathbf{b}$, as test functions in (5.1.80) and (5.1.81), respectively, yields (5.1.76). In an analogous way, we substitute the functions $\mathbf{v}x_2$, $\mathbf{v}x_3$, $\mathbf{v}x_2^2/2$, $\mathbf{v}x_2x_3$ and $\mathbf{v}x_2x_3$, $\mathbf{v}x_3^2/2$, where $\mathbf{v} = v\mathbf{n}$ or $\mathbf{v} = v\mathbf{b}$, to (5.1.80) and (5.1.81), respectively, to derive (5.1.77). □

If we define the vector $\boldsymbol{\eta} \in L^2(\Omega)^2$ by $\boldsymbol{\eta} := (\zeta_{12}, \zeta_{13})$, then the equations (5.1.70) after putting $\mathbf{v} = \varphi\mathbf{t}$, $\varphi \in L^2(0,l;W^{1,2}(S))$, and (5.1.31) can be rewritten in the form

$$\int_\Omega \boldsymbol{\eta} \cdot \nabla_{23}\varphi \, dx = 0, \ \forall \varphi \in L^2(0,l;W^{1,2}(S)), \tag{5.1.82}$$

$$\int_\Omega \boldsymbol{\eta} \cdot \mathrm{rot}_{23}\psi \, dx = \int_\Omega \mathbf{u}'_* \cdot \mathbf{t}\psi \, dx, \ \forall \psi \in W_0^{1,2}(\Omega), \tag{5.1.83}$$

where we have denoted $\nabla_{23}\varphi := (\partial_2\varphi, \partial_3\varphi)$, $\mathrm{rot}_{23}\psi := (-\partial_3\psi, \partial_2\psi)$ and the scalar product here means the scalar product in the usual two dimensional Euclidean space $\mathbb{R}^2$.

Lemma 5.1.10 *Let S be a simply connected domain and let $\partial S \in C^1$. The system* (5.1.82), (5.1.83) *has a unique solution in $L^2(\Omega)^2$, given by*

$$\boldsymbol{\eta} = (\zeta_{12}, \zeta_{13}) = -\frac{1}{2}(\mathbf{u}'_* \cdot \mathbf{t})(\partial_2 p - x_3, \partial_3 p + x_2), \tag{5.1.84}$$

where the function $p \in W^{1,2}(S)$ is the unique solution to the Neumann problem

$$\int_S [(\partial_2 p - x_3)\partial_2 r + (\partial_3 p + x_2)\partial_3 r] \, dx_2 dx_3 = 0, \ \int_S p \, dx_2 dx_3 = 0, \tag{5.1.85}$$

for all $r \in W^{1,2}(S)$.

Proof: After substitution of (5.1.84) to (5.1.82) and (5.1.83), we obtain, using (5.1.85), that

$$\int_\Omega \boldsymbol{\eta} \cdot \nabla_{23}\varphi \, dx = -\frac{1}{2}\int_\Omega (\mathbf{u}'_* \cdot \mathbf{t})(\partial_2 p - x_3)\partial_2\varphi \, dx - \frac{1}{2}\int_\Omega (\mathbf{u}'_* \cdot \mathbf{t})(\partial_3 p + x_2)\partial_3\varphi \, dx =$$

$$= -\frac{1}{2}\int_0^l (\mathbf{u}'_* \cdot \mathbf{t}) \int_S [(\partial_2 p - x_3)\partial_2\varphi + (\partial_3 p + x_2)\partial_3\varphi] \, dx_2 dx_3 dx_1 \overset{(5.1.85)}{=} 0$$

and

$$\int_\Omega \boldsymbol{\eta} \cdot \mathrm{rot}_{23}\psi \, dx = \frac{1}{2}\int_\Omega (\mathbf{u}'_* \cdot \mathbf{t})(\partial_2 p - x_3)\partial_3\psi \, dx - \frac{1}{2}\int_\Omega (\mathbf{u}'_* \cdot \mathbf{t})(\partial_3 p + x_2)\partial_2\psi \, dx =$$

$$= -\frac{1}{2}\int_0^l (\mathbf{u}'_* \cdot \mathbf{t}) \left[\int_S \partial_3 p \partial_2\psi - \partial_2 p \partial_3\psi \, dx_2 dx_3 + \int_S x_3\partial_3\psi + x_2\partial_2\psi \, dx_2 dx_3\right] dx_1 =$$

$$= \int_\Omega \mathbf{u}'_* \cdot \mathbf{t}\psi \, dx,$$

for all $\psi \in C_0^\infty(\Omega)$, which implies that $\psi(x_1, \cdot, \cdot) \in C_0^\infty(S)$ for all $x_1 \in (0,l)$. Thus by density the relation remains valid for all $\psi \in W_0^{1,2}(\Omega)$.

To prove uniqueness, we assume that there exist two solutions $\boldsymbol{\eta}_i \in L^2(\Omega)^2$, $i = 1, 2$. Taking $\varphi = s\widehat{\varphi}$ in (5.1.82) and $\psi = s\widehat{\psi}$ in (5.1.83) for all $s \in C_0^\infty(0,l)$, $\widehat{\varphi} \in W^{1,2}(S)$ and $\widehat{\psi} \in W_0^{1,2}(S)$, it is easy to verify that the function $\boldsymbol{\eta}_s := (\eta_{1,s}, \eta_{2,s}) = \int_0^l s\boldsymbol{\eta} \, dx_1$, where $\boldsymbol{\eta} = \boldsymbol{\eta}_1 - \boldsymbol{\eta}_2$, satisfies the equations

$$\int_S \boldsymbol{\eta}_s \cdot \nabla_{23}\widehat{\varphi} \, dx_2 dx_3 = 0 \text{ and } \int_S \boldsymbol{\eta}_s \cdot \mathrm{rot}_{23}\widehat{\psi} \, dx_2 dx_3 = 0. \tag{5.1.86}$$

Let us define the vector functions $\widehat{\boldsymbol{\eta}}_s = (0, \eta_{1,s}, \eta_{2,s})$ and $\widehat{\boldsymbol{\psi}} = (-\check{\psi}, \psi_1, \psi_2)$, where the functions $\check{\psi},\ \psi_1,\ \psi_2 \in C_0^\infty(\Omega)$ are arbitrary. Since the function $\widehat{\boldsymbol{\eta}}_s$ is defined only on S, we can deduce from (5.1.86) that

$$\int_0^l \int_S \widehat{\boldsymbol{\eta}}_s \cdot \operatorname{rot} \widehat{\boldsymbol{\psi}}\ dx = \int_0^l \int_S \boldsymbol{\eta}_s \cdot \operatorname{rot}_{23} \check{\psi}(x_1)\ dx_2 dx_3 dx_1 = 0.$$

Hence, we can easily derive that $\operatorname{rot} \widehat{\boldsymbol{\eta}}_s = 0$ in $\mathcal{D}'(\Omega)$. Since S is simply connected, then $\Omega = [0,l] \times S$ is simply connected as well and there exists a function $h_s \in W^{1,2}(\Omega)$, unique up to a constant, such that $\widehat{\boldsymbol{\eta}}_s = \nabla h_s$ (see [83]), which means

$$\partial_1 h_s = 0,\ \partial_2 h_s = \eta_{1,s},\ \partial_3 h_s = \eta_{2,s},$$

and hence we get that $h_s \in W^{1,2}(S)$ and $\boldsymbol{\eta}_s = \nabla_{23} h_s$. After substitution $\widehat{\varphi} = h_s$ to (5.1.86), it follows that $\|\nabla_{23} h_s\|_2 = 0$. Hence $\boldsymbol{\eta}_s = \mathbf{0}$ for all $s \in L^2(0,l)$, which implies $\boldsymbol{\eta} = \mathbf{0}$. □

5.1.4 A priori estimates, related convergences and properties of limits

The standard technique in partial differential equations is the derivation of a priori estimates. One of their important consequences is the option to derive various weak convergences. In the thesis, the weak convergences are used in the derivation of our limit equations. The derivation of the a priori estimates is however closely related to the existence of a solution to (5.1.18) and (5.1.19). We omit the proof of existence because this is not related to our main topic. The proof can be done similarly as in [65] and [75]. For more details and comments we refer the reader to [210].

If we summarize the results from [210] we can prove the existence of a solution $\mathbf{u}_\epsilon$ to the problem (5.1.18)–(5.1.19) such that $\mathbf{u}_\epsilon \in L^\infty(0,T;V(\Omega))$, $\partial_t \mathbf{u}_\epsilon \in L^\infty(0,T;L^2(\Omega)^3)$, $\rho \partial_{tt} \mathbf{u}_\epsilon \in L^2(0,T;[V(\Omega)]')$, where the initial conditions in (5.1.19) are fulfilled in the sense of the spaces $C([0,T];L^2(\Omega)^3)$ and $C([0,T];L^2_{weak}(\Omega)^3)$, respectively. In addition, this solution satisfies for all $\epsilon \in (0,1)$ the estimates

$$\|\partial_t \mathbf{u}_\epsilon\|^2_{L^\infty(0,T;L^2(\Omega)^3)} + \left\| \frac{1}{\epsilon} \omega(\mathbf{u}_\epsilon) \right\|^2_{L^\infty(0,T;L^2(\Omega)^9)} \le C\Big(\|\mathbf{q}_{1,\epsilon}\|_2^2 +$$

$$+ \left\| \frac{1}{\epsilon} \omega(\mathbf{q}_{0,\epsilon}) \right\|_2^2 + \|\mathbf{f}\|^2_{L^2(0,T;L^2(\Omega)^3)} + \|\mathbf{h}\|^2_{W^{1,1}(0,T;L^2(0,l;L^2(\partial S)^3))} \Big) \tag{5.1.87}$$

and

$$\|\rho \partial_{tt} \mathbf{u}_\epsilon\|_{L^2(0,T;[V(\Omega)]')} \le C\Big(\|\mathbf{f}\|_{L^2(0,T;L^2(\Omega)^3)} +$$

$$+ \|\mathbf{h}\|_{L^2(0,T;L^2(0,l;L^2(\partial S)^3))} + \frac{1}{\epsilon^2} \|\omega^\epsilon(\mathbf{u}_\epsilon)\|_{L^2(0,T;L^2(\Omega)^9)} \Big), \tag{5.1.88}$$

where the constant C is independent of ϵ. Using the a priori estimates together with Proposition 5.1.6 we can prove the following corollary.

Corollary 5.1.11 *It follows from* (5.1.87) *that there exists a sequence* $\{\epsilon_n\}_{n=1}^\infty \subset (0,1)$ *such that* $\epsilon_n \to 0$ *and*

$$\mathbf{u}_{\epsilon_n} \overset{*}{\rightharpoonup} \mathbf{u} \text{ in } L^\infty(0,T;V(\Omega)), \tag{5.1.89}$$

$$\partial_t \mathbf{u}_{\epsilon_n} \overset{*}{\rightharpoonup} \partial_t \mathbf{u} \text{ in } L^\infty(0,T;L^2(\Omega)^3), \tag{5.1.90}$$

$$\frac{1}{\epsilon_n} \omega^{\epsilon_n}(\mathbf{u}_{\epsilon_n}) \overset{*}{\rightharpoonup} \zeta \text{ in } L^\infty(0,T;L^2(\Omega)^9) \tag{5.1.91}$$

and thus

$$\overline{\mathbf{u}_{\epsilon_n}}^{\varphi} \rightharpoonup \overline{\mathbf{u}}^{\varphi} \text{ in } W^{1,2}(\Omega)^3, \tag{5.1.92}$$

$$\overline{\partial_t \mathbf{u}_{\epsilon_n}}^{\varphi} \rightharpoonup \overline{\partial_t \mathbf{u}}^{\varphi} \text{ in } L^2(\Omega)^3, \tag{5.1.93}$$

$$\frac{1}{\epsilon_n}\omega^{\epsilon_n}(\overline{\mathbf{u}_{\epsilon_n}}^{\varphi}) = \overline{\frac{1}{\epsilon_n}\omega^{\epsilon_n}(\mathbf{u}_{\epsilon_n})}^{\varphi} \rightharpoonup \overline{\zeta}^{\varphi} \text{ in } L^2(\Omega)^9 \tag{5.1.94}$$

for all $\varphi \in C_0^\infty(0,T)$, *where*

$$\overline{u}^{\varphi}(x) := \int_0^T \varphi(t)u(x,t)\, dt.$$

Using the corollary together with Proposition 5.1.6 we can easily check the following proposition.

Proposition 5.1.12 *Suppose that* $\{\epsilon_n\}_{n=1}^\infty \subset (0,1)$ *and* $\epsilon_n \to 0$. *Let, in addition, a sequence* $\{\mathbf{u}_{\epsilon_n}\}_{n=1}^\infty \subset L^\infty(0,T;V(\Omega))$ *be such that*

$$\mathbf{u}_{\epsilon_n} \overset{*}{\rightharpoonup} \mathbf{u} \text{ in } L^\infty(0,T;V(\Omega)),$$
$$\frac{1}{\epsilon_n}\omega^{\epsilon_n}(\mathbf{u}_{\epsilon_n}) \overset{*}{\rightharpoonup} \zeta \text{ in } L^\infty(0,T;L^2(\Omega)^9)$$

for $\epsilon_n \to 0$. *Then the couple* $\langle \mathbf{u}, \phi\rangle \in L^\infty(0,T;\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l))$ *(in the sense* $\partial_j \mathbf{u} = 0$, $j = 2,3$*), where the function* ϕ *is such that*

$$\frac{1}{2\epsilon_n}\left(\partial_2 \mathbf{u}_{\epsilon_n}\cdot \mathbf{b}_{\epsilon_n} - \partial_3 \mathbf{u}_{\epsilon_n}\cdot \mathbf{n}_{\epsilon_n}\right) \overset{*}{\rightharpoonup} \phi \tag{5.1.95}$$

in $L^\infty(0,T;L^2(\Omega))$ *for* $\epsilon_n \to 0$. *In addition, the couple* $\langle \mathbf{u}, \phi\rangle$ *generates the function* $\mathbf{u}_* \in L^\infty(0,T;W_0^{1,2}(0,l)^3)$ *which together with the function* $\mathbf{u}$ *satisfy the relations*

$$\partial_1 \mathbf{u}\cdot \mathbf{t} = 0 \text{ a.e. in } (0,l)\times(0,T), \tag{5.1.96}$$

$$\partial_1 \mathbf{u}_*\cdot \mathbf{t} = \partial_3\zeta_{12} - \partial_2\zeta_{13} \text{ in } L^\infty(0,T;L^2(0,l;[W_0^{1,2}(S)]')), \tag{5.1.97}$$

$$\partial_1 \mathbf{u}_*\cdot \mathbf{n} = -\partial_3\zeta_{11} \text{ a.e. in } (0,l)\times(0,T), \tag{5.1.98}$$

$$\partial_1 \mathbf{u}_*\cdot \mathbf{b} = \partial_2\zeta_{11} \text{ a.e. in } (0,l)\times(0,T). \tag{5.1.99}$$

Remark 5.1.13 Since $\overline{\frac{1}{\epsilon_n}\omega^{\epsilon_n}(\mathbf{u}_{\epsilon_n})}^{\varphi} = \frac{1}{\epsilon_n}\omega^{\epsilon_n}(\overline{\mathbf{u}_{\epsilon_n}}^{\varphi})$ (see (5.1.5)–(5.1.8)), we can use (5.1.92), (5.1.94) and Proposition 5.1.6 to derive the existence of the pair $\langle \overline{\mathbf{u}}^{\varphi}, \phi_\varphi\rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ (in the sense $\partial_j \overline{\mathbf{u}}^{\varphi} = 0$, $j = 2,3$) for arbitrary $\varphi \in C_0^\infty(0,T)$, where the function ϕ_φ is such that

$$\frac{1}{2\epsilon_n}\left(\partial_2 \overline{\mathbf{u}_{\epsilon_n}}^{\varphi}\cdot \mathbf{b}_{\epsilon_n} - \partial_3 \overline{\mathbf{u}_{\epsilon_n}}^{\varphi}\cdot \mathbf{n}_{\epsilon_n}\right) \rightharpoonup \phi_\varphi \tag{5.1.100}$$

in $L^2(\Omega)$ for $\epsilon_n \to 0$ and for arbitrary $\varphi \in C_0^\infty(0,T)$. In addition, the couple $\langle \overline{\mathbf{u}}^{\varphi}, \phi_\varphi\rangle$ generates the function $\mathbf{u}_{*,\varphi} \in W_0^{1,2}(0,l)^3$ which together with the function $\overline{\mathbf{u}}^{\varphi}$ satisfy the relations

$$\partial_1 \overline{\mathbf{u}}^{\varphi}\cdot \mathbf{t} = 0 \text{ a.e. in } (0,l), \tag{5.1.101}$$

$$\partial_1 \mathbf{u}_{*,\varphi}\cdot \mathbf{t} = \overline{\partial_3\zeta_{12} - \partial_2\zeta_{13}}^{\varphi} \text{ in } L^2(0,l;[W_0^{1,2}(S)]'), \tag{5.1.102}$$

$$\partial_1 \mathbf{u}_{*,\varphi}\cdot \mathbf{n} = \overline{-\partial_3\zeta_{11}}^{\varphi} \text{ a.e. in } (0,l), \tag{5.1.103}$$

$$\partial_1 \mathbf{u}_{*,\varphi}\cdot \mathbf{b} = \overline{\partial_2\zeta_{11}}^{\varphi} \text{ a.e. in } (0,l) \tag{5.1.104}$$

for arbitrary $\varphi \in C_0^\infty(0,T)$. If the sequence $\{\frac{1}{\epsilon_n}\omega^{\epsilon_n}(\overline{\mathbf{u}_{\epsilon_n}}^{\varphi})\}_{n=1}^\infty$ converges strongly in $L^2(\Omega)^9$, the convergence of the sequence $\{\overline{\mathbf{u}_{\epsilon_n}}^{\varphi}\}_{n=1}^\infty$ is strong as well for arbitrary $\varphi \in C_0^\infty(0,T)$.

5.1.5 Proof of the main result

In the section, we prove the main result formulated in Theorem 5.1.1. The proof is decomposed into several lemmas to cover various parts of the theorem.

First, let us introduce some constants we use in the limit equation (5.1.13):

$$I_1 := \int_S x_2^2 \; dx_2 dx_3, \; I_2 := \int_S x_3^2 \; dx_2 dx_3, \tag{5.1.105}$$

$$E := \mu \frac{3\lambda + 2\mu}{\lambda + \mu}, \; K := \int_S [(\partial_2 p - x_3)^2 + (\partial_3 p + x_2)^2] \; dx_2 dx_3, \tag{5.1.106}$$

where $p \in W^{1,2}(S)$ is the unique solution to the Neumann problem (5.1.85).

Lemma 5.1.14 *Let* $\{\mathbf{u}_{\epsilon_n}\}_{n=1}^{\infty}$*,* $\epsilon_n \to 0$*, be a subsequence of the weak solutions to the problem* (5.1.18)–(5.1.19) *satisfying* (5.1.87), (5.1.89)–(5.1.91)*. Then the limit* $\langle \mathbf{u}, \phi \rangle \in L^\infty(0,T; \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l))$ *obtained in Proposition* 5.1.6 *generates the function* $\mathbf{u}_*$ *that satisfies the equation*

$$-\int_0^l \check{\rho}\overline{\partial_t \mathbf{u}}^{\dot{\varphi}} \cdot \mathbf{v} \; dx_1 + \int_0^l E[I_1(\overline{\partial_1 \mathbf{u}_*}^{\varphi} \cdot \mathbf{b})(\mathbf{v}_*' \cdot \mathbf{b}) + I_2(\overline{\partial_1 \mathbf{u}_*}^{\varphi} \cdot \mathbf{n})(\mathbf{v}_*' \cdot \mathbf{n})] \; dx_1 +$$

$$+ \int_0^l \mu K(\overline{\partial_1 \mathbf{u}_*}^{\varphi} \cdot \mathbf{t})(\mathbf{v}_*' \cdot \mathbf{t}) \; dx_1 = \int_0^l (\overline{\check{\mathbf{f}}_{\mathbf{f}+\mathbf{h}}}^{\varphi} \cdot \mathbf{v}) \; dx_1 \tag{5.1.107}$$

for all functions $\mathbf{v}_* \in W_0^{1,2}(0,l)^3$ *generated by the respective couples* $\langle \mathbf{v}, \psi \rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ *and for all functions* $\varphi \in C_0^\infty(0,T)$*. In* (5.1.107) *we use the notation* $\check{\rho}(x_1) := \int_S \rho(x_1, x_2, x_3) \; dx_2 dx_3$ *and* $\check{\mathbf{f}}_{\mathbf{f}+\mathbf{h}}(x_1, t) := \int_S \mathbf{f}(x_1, x_2, x_3, t) \; dx_2 dx_3 + \int_{\partial S} \mathbf{h}(x_1, x_2, x_3, t) \; dS$*,* $(x_1, t) \in (0,l) \times (0,T)$*.*

Proof: Let us use ϵ instead of ϵ_n to simplify notation. Let $\langle \mathbf{v}, \psi \rangle$ be an arbitrary couple of functions from the space $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ and the couples $\langle \mathbf{v}_\epsilon, \psi_\epsilon \rangle \in \mathcal{V}_0^{\mathbf{t}_\epsilon,\mathbf{n}_\epsilon,\mathbf{b}_\epsilon}(0,l)$ its smooth approximations given by Proposition 4.4.2. We define the functions $\mathbf{w}_\epsilon \in C^\infty(\overline{\Omega})^3$ and $\widehat{\mathbf{v}}_\epsilon \in C^\infty(\overline{\Omega})^3 \cap V(\Omega)$ by

$$\begin{aligned}
\mathbf{w}_\epsilon(x_1, x_2, x_3) \;\; &:= \;\; -\Big((\mathbf{v}_\epsilon'(x_1) \cdot \mathbf{n}_\epsilon(x_1))x_2 + (\mathbf{v}_\epsilon'(x_1) \cdot \mathbf{b}_\epsilon(x_1))x_3\Big)\mathbf{t}_\epsilon(x_1) - \\
&\;\;\;\; -x_3\psi_\epsilon(x_1)\mathbf{n}_\epsilon(x_1) + x_2\psi_\epsilon(x_1)\mathbf{b}_\epsilon(x_1), \\
\widehat{\mathbf{v}}_\epsilon(x_1, x_2, x_3) \;\; &:= \;\; \mathbf{v}_\epsilon(x_1) + \epsilon\mathbf{w}_\epsilon(x_1, x_2, x_3) \text{ for } (x_1, x_2, x_3) \in \Omega.
\end{aligned}$$

We can put $\widehat{\mathbf{v}}_\epsilon$ to (5.1.5)–(5.1.8) and using (4.2.3), (4.2.6), (5.1.41) we can verify (see Lemma 8.4 in [199] for more details) that

$$\omega^\epsilon(\widehat{\mathbf{v}}_\epsilon) = \epsilon\Upsilon(\mathbf{v}_{*,\epsilon}) + B_\epsilon, \tag{5.1.108}$$

where

$$\begin{aligned}
\Upsilon_{11}(\mathbf{v}_{*,\epsilon}) \;\; &= \;\; -(\mathbf{v}_{*,\epsilon}' \cdot \mathbf{n}_\epsilon)x_3 + (\mathbf{v}_{*,\epsilon}' \cdot \mathbf{b}_\epsilon)x_2, && (5.1.109) \\
\Upsilon_{12}(\mathbf{v}_{*,\epsilon}) \;\; &= \;\; \Upsilon_{21}(\mathbf{v}_{*,\epsilon}) = \frac{x_3}{2}(\mathbf{v}_{*,\epsilon}' \cdot \mathbf{t}_\epsilon), && (5.1.110) \\
\Upsilon_{13}(\mathbf{v}_{*,\epsilon}) \;\; &= \;\; \Upsilon_{31}(\mathbf{v}_{*,\epsilon}) = -\frac{x_2}{2}(\mathbf{v}_{*,\epsilon}' \cdot \mathbf{t}_\epsilon), && (5.1.111) \\
\Upsilon_{ij}(\mathbf{v}_{*,\epsilon}) \;\; &= \;\; 0, \; i,j = 2,3, && (5.1.112)
\end{aligned}$$

$$\begin{aligned}
B_\epsilon^{11} = \epsilon^2\Big(&(\beta_\epsilon x_2 + \alpha_\epsilon x_3)(x_2(\mathbf{v}_\epsilon' \cdot \mathbf{n}_\epsilon)' + x_3(\mathbf{v}_\epsilon' \cdot \mathbf{b}_\epsilon)' - \beta_\epsilon x_3\psi_\epsilon + \alpha_\epsilon x_2\psi_\epsilon) + \\
&+ \gamma_\epsilon x_3(\partial_1\mathbf{w}_\epsilon \cdot \mathbf{n}_\epsilon) - \gamma_\epsilon x_2(\partial_1\mathbf{w}_\epsilon \cdot \mathbf{b}_\epsilon)\Big),
\end{aligned} \tag{5.1.113}$$

$B_\epsilon^{ij} = 0$ for $i, j \neq 1$.

Further, we have from (4.2.3), Proposition 4.3.2 and Proposition 4.4.2 that

$$\Upsilon_{ij}(\mathbf{v}_{*,\epsilon}) \to \Upsilon_{ij}(\mathbf{v}_*) \text{ in } L^2(\Omega),\ i, j = 1, 2, 3,$$

$$\|B_\epsilon\|_2 = \|B_\epsilon^{11}\|_2 \leq C\epsilon^{2(1-r)},\ r \in \left(0, \frac{1}{3}\right),$$

$$\widehat{\mathbf{v}}_\epsilon \to \mathbf{v} \text{ in } W^{1,2}(\Omega)^3$$

for $\epsilon \to 0$.

It follows from the convergences (5.1.25), (5.1.26), (5.1.73) (5.1.89)–(5.1.91) and the convergences for $\widehat{\mathbf{v}}_\epsilon$ above that we can pass from the equation

$$-\int_\Omega \rho\overline{\partial_t \mathbf{u}_\epsilon}^{\dot{\varphi}} \cdot \widehat{\mathbf{v}}_\epsilon d_\epsilon \; dx + \int_\Omega A_\epsilon^{ijkl} \frac{1}{\epsilon}\omega_{kl}^\epsilon(\overline{\mathbf{u}_\epsilon}^\varphi) \frac{1}{\epsilon}\omega_{ij}^\epsilon(\widehat{\mathbf{v}}_\epsilon) d_\epsilon \; dx = \int_\Omega \overline{\mathbf{f}}^\varphi \cdot \widehat{\mathbf{v}}_\epsilon d_\epsilon \; dx +$$
$$+ \int_0^l \int_{\partial S} \overline{\mathbf{h}}^\varphi \cdot \widehat{\mathbf{v}}_\epsilon d_\epsilon \epsilon \sqrt{\nu_j o^{ij,\epsilon} \nu_j} \; dS dx_1$$

to

$$-\int_0^l \check{\rho}\overline{\partial_t \mathbf{u}}^{\dot{\varphi}} \cdot \mathbf{v} \; dx_1 + \int_\Omega A_0^{ijkl}\overline{\zeta_{kl}}^\varphi \Upsilon_{ij}(\mathbf{v}_*) \; dx = \int_\Omega \overline{\mathbf{f}}^\varphi \cdot \mathbf{v} \; dx +$$
$$+ \int_0^l \int_{\partial S} \overline{\mathbf{h}}^\varphi \cdot \mathbf{v} \; dS dx_1 \tag{5.1.114}$$

for all functions $\langle \mathbf{v}, \psi \rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0, l)$ generating functions $\mathbf{v}_*$.

It remains to express the second term in (5.1.114). Equalities (5.1.32) and (5.1.33) enable us to express the function ζ_{11} in this way

$$\zeta_{11} = Q_0 + (\partial_1 \mathbf{u}_* \cdot \mathbf{b})x_2 - (\partial_1 \mathbf{u}_* \cdot \mathbf{n})x_3 \text{ in } \Omega \times (0, T). \tag{5.1.115}$$

Hence and from (4.2.1), (5.1.73), (5.1.74)–(5.1.77), (5.1.84), and (5.1.115), we can conclude that

$$\int_\Omega A_0^{ijkl}\overline{\zeta_{kl}}^\varphi \Upsilon_{ij}(\mathbf{v}_*) \; dx = \int_\Omega [\lambda(\overline{\zeta_{11}}^\varphi + \overline{\zeta_{22}}^\varphi + \overline{\zeta_{33}}^\varphi) + 2\mu\overline{\zeta_{11}}^\varphi]\Upsilon_{11}(\mathbf{v}_*) \; dx +$$
$$+ \int_\Omega [4\mu(\overline{\zeta_{12}}^\varphi \Upsilon_{12}(\mathbf{v}_*) + \overline{\zeta_{13}}^\varphi \Upsilon_{13}(\mathbf{v}_*))] \; dx =$$
$$= \int_\Omega [\lambda(\overline{\zeta_{11}}^\varphi + \overline{\zeta_{22}}^\varphi + \overline{\zeta_{33}}^\varphi) + 2\mu\overline{\zeta_{11}}^\varphi][(\mathbf{v}_*' \cdot \mathbf{b})x_2 - (\mathbf{v}_*' \cdot \mathbf{n})x_3] \; dx +$$
$$+2\mu \int_\Omega [\overline{\zeta_{12}}^\varphi (\mathbf{v}_*' \cdot \mathbf{t})x_3 - \overline{\zeta_{13}}^\varphi (\mathbf{v}_*' \cdot \mathbf{t})x_2] \; dx =$$
$$= \int_\Omega [E\overline{\zeta_{11}}^\varphi + \lambda(\overline{\zeta_{22}^H}^\varphi + \overline{\zeta_{33}^H}^\varphi)][(\mathbf{v}_*' \cdot \mathbf{b})x_2 - (\mathbf{v}_*' \cdot \mathbf{n})x_3] \; dx +$$
$$+ \int_\Omega \mu\left(-(\partial_2 p - x_3)x_3 + (\partial_3 p + x_2)x_2\right)(\partial_1 \overline{\mathbf{u}_*}^\varphi \cdot \mathbf{t})(\mathbf{v}_*' \cdot \mathbf{t}) \; dx =$$
$$= \int_0^l E[I_1(\overline{\partial_1 \mathbf{u}_*}^\varphi \cdot \mathbf{b})(\mathbf{v}_*' \cdot \mathbf{b}) + I_2(\partial_1 \overline{\mathbf{u}_*}^\varphi \cdot \mathbf{n})(\mathbf{v}_*' \cdot \mathbf{n})] \; dx_1 +$$
$$+ \int_0^l \mu K(\overline{\partial_1 \mathbf{u}_*}^\varphi \cdot \mathbf{t})(\mathbf{v}_*' \cdot \mathbf{t}) \; dx_1. \tag{5.1.116}$$

□

Lemma 5.1.15 *It holds* $Q_0 = \zeta_{22}^H = \zeta_{23}^H = \zeta_{33}^H = 0$ *in* $\Omega \times (0,T)$.

Proof: According to Proposition 5.1.2 and (5.1.25), there exists a constant $C > 0$ independent of ϵ and φ such that

$$\left\| \frac{1}{\epsilon}\omega^\epsilon(\overline{\mathbf{u}_\epsilon}^\varphi) - \overline{\zeta}^\varphi \right\|_2^2 \leq C\Lambda_{\epsilon,\varphi} \tag{5.1.117}$$

for all $\varphi \in C_0^\infty(0,T)$, where

$$\Lambda_{\epsilon,\varphi} := \int_\Omega A_\epsilon^{ijkl} \left(\frac{1}{\epsilon}\omega_{kl}^\epsilon(\overline{\mathbf{u}_\epsilon}^\varphi) - \overline{\zeta_{kl}}^\varphi \right) \left(\frac{1}{\epsilon}\omega_{ij}^\epsilon(\overline{\mathbf{u}_\epsilon}^\varphi) - \overline{\zeta_{ij}}^\varphi \right) d_\epsilon \; dx.$$

Convergences (5.1.89)–(5.1.91) and equation (5.1.18) imply that

$$\begin{aligned} \Lambda_\varphi = \lim_{\epsilon\to 0} \Lambda_{\epsilon,\varphi} = \lim_{\epsilon\to 0} \Bigg[\int_\Omega \overline{\mathbf{f}}^\varphi \cdot \overline{\mathbf{u}_\epsilon}^\varphi d_\epsilon \; dx + \int_0^l \int_{\partial S} \overline{\mathbf{h}}^\varphi \cdot \overline{\mathbf{u}_\epsilon}^\varphi \; d_\epsilon \epsilon \sqrt{\nu_i o^{ij,\epsilon}\nu_j} \; dS dx_1 + \\ + \int_\Omega A_\epsilon^{ijkl} \left(\left(\overline{\zeta_{kl}}^\varphi - \frac{1}{\epsilon}\omega_{kl}^\epsilon(\overline{\mathbf{u}_\epsilon}^\varphi) \right) \overline{\zeta_{ij}}^\varphi - \overline{\zeta_{kl}}^\varphi \frac{1}{\epsilon}\omega_{ij}^\epsilon(\overline{\mathbf{u}_\epsilon}^\varphi) \right) d_\epsilon \; dx + \\ + \int_\Omega \rho \overline{\partial_t \mathbf{u}_\epsilon}^{\dot\varphi} \cdot \overline{\mathbf{u}_\epsilon}^\varphi \; d_\epsilon \; dx \Bigg] = \int_0^l \overline{\check{\mathbf{f}}_{\mathbf{f}+\mathbf{h}}}^\varphi \cdot \overline{\mathbf{u}}^\varphi \; dx_1 - \\ - \int_\Omega A_0^{ijkl} \overline{\zeta_{kl}}^\varphi \overline{\zeta_{ij}}^\varphi \; dx + \int_0^l \check\rho \overline{\partial_t \mathbf{u}}^{\dot\varphi} \cdot \overline{\mathbf{u}}^\varphi \; dx_1. \end{aligned} \tag{5.1.118}$$

Using (5.1.74), (5.1.84), (5.1.107), and (5.1.115) we get analogously as in the proof of Lemma 8.5 in [199] that

$$\begin{aligned} \int_\Omega A_0^{ijkl} \overline{\zeta_{kl}}^\varphi \overline{\zeta_{ij}}^\varphi \; dx = \int_\Omega \Bigg[E\Big(\overline{Q_0}^\varphi + (\overline{\partial_1 \mathbf{u}_*}^\varphi \cdot \mathbf{b})x_2 - (\overline{\partial_1 \mathbf{u}_*}^\varphi \cdot \mathbf{n})x_3\Big)^2 + \\ + 4\mu \left(-\frac{1}{2}(\overline{\partial_1 \mathbf{u}_*}^\varphi \cdot \mathbf{t})(\partial_2 p - x_3) \right)^2 + 4\mu \left(-\frac{1}{2}(\overline{\partial_1 \mathbf{u}_*}^\varphi \cdot \mathbf{t})(\partial_3 p + x_2) \right)^2 + \\ + \lambda(\overline{\zeta_{22}^H}^\varphi + \overline{\zeta_{33}^H}^\varphi)^2 + 2\mu((\overline{\zeta_{22}^H}^\varphi)^2 + (\overline{\zeta_{33}^H}^\varphi)^2 + 2(\overline{\zeta_{23}^H}^\varphi)^2) \Bigg] \; dx = \\ = \int_0^l \left[\overline{\check{\mathbf{f}}_{\mathbf{f}+\mathbf{h}}}^\varphi \cdot \overline{\mathbf{u}}^\varphi + E|S|(\overline{Q_0}^\varphi)^2 \right] \; dx_1 + \\ + \int_0^l \check\rho \overline{\partial_t \mathbf{u}}^{\dot\varphi} \cdot \overline{\mathbf{u}}^\varphi \; dx_1 + \int_\Omega \left[\lambda(\overline{\zeta_{22}^H}^\varphi + \overline{\zeta_{33}^H}^\varphi)^2 + 2\mu \left((\overline{\zeta_{22}^H}^\varphi)^2 + (\overline{\zeta_{33}^H}^\varphi)^2 + 2(\overline{\zeta_{23}^H}^\varphi)^2 \right) \right] \; dx. \end{aligned}$$

After substitution to (5.1.117), we obtain

$$\Lambda_\varphi = -\int_\Omega \left[E(\overline{Q_0}^\varphi)^2 + \lambda(\overline{\zeta_{22}^H}^\varphi + \overline{\zeta_{33}^H}^\varphi)^2 + 2\mu \left((\overline{\zeta_{22}^H}^\varphi)^2 + (\overline{\zeta_{33}^H}^\varphi)^2 + 2(\overline{\zeta_{23}^H}^\varphi)^2 \right) \right] \; dx$$

for all $\varphi \in C_0^\infty(0,T)$. But the sequence $\Lambda_{\epsilon,\varphi}$ for all $\varphi \in C_0^\infty(0,T)$ consists of non-negative numbers by (5.1.118) and thus $\Lambda_\varphi = 0$ for all $\varphi \in C_0^\infty(0,T)$. □

Since we have denoted $\boldsymbol{\eta} = (\zeta_{12}, \zeta_{13})$, we obtain from Lemma 5.1.15 that

$$\zeta_{11} \overset{(5.1.115)}{=} (\partial_1 \mathbf{u}_* \cdot \mathbf{b})x_2 - (\partial_1 \mathbf{u}_* \cdot \mathbf{n})x_3, \; \zeta_{12} \overset{(5.1.84)}{=} \zeta_{21} = -\frac{1}{2}(\partial_1 \mathbf{u}_* \cdot \mathbf{t})(\partial_2 p - x_3),$$

$$\zeta_{13} \overset{(5.1.84)}{=} \zeta_{31} = -\frac{1}{2}(\partial_1 \mathbf{u}_* \cdot \mathbf{t})(\partial_3 p + x_2),$$

$$\zeta_{22} \overset{(5.1.74)}{=} -\frac{1}{2}\frac{\lambda}{\lambda+\mu}\Big((\partial_1 \mathbf{u}_* \cdot \mathbf{b})x_2 - (\partial_1 \mathbf{u}_* \cdot \mathbf{n})x_3\Big), \; \zeta_{23} = \zeta_{32} = 0,$$

$$\zeta_{33} \overset{(5.1.74)}{=} -\frac{1}{2}\frac{\lambda}{\lambda+\mu}\Big((\partial_1 \mathbf{u}_* \cdot \mathbf{b})x_2 - (\partial_1 \mathbf{u}_* \cdot \mathbf{n})x_3\Big).$$

Lemma 5.1.16 *Let the function* $\mathbf{u}$ *be determined by* (5.1.89) *and the functions* $\mathbf{q}_0$ *and* $\mathbf{q}_1$ *by* (5.1.17). *Then* $\mathbf{u}|_{t=0} = \mathbf{q}_0$ *and* $\check{\rho}\partial_t \mathbf{u}|_{t=0} = \check{\rho}\mathbf{q}_1$ *in the sense of the space* $C([0,T]; L^2(\Omega)^3)$ *or* $C([0,T]; [\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)]')$*, respectively.*

Proof: The first initial condition follows easily from (5.1.17), (5.1.19), (5.1.89), and (5.1.90). Let the functions $\widehat{\mathbf{v}}_\epsilon$ be the same as in the proof of Lemma 5.1.14 and let $\varphi \in C_0^\infty(0,T)$ be an arbitrary but fixed function. Taking $\varphi\widehat{\mathbf{v}}_\epsilon$ as the test functions in (5.1.18) and using (5.1.108)–(5.1.113) lead to the equation

$$-\int_0^T \dot{\varphi}(t) \int_\Omega \rho \partial_t \mathbf{u}_\epsilon(t) \cdot \widehat{\mathbf{v}}_\epsilon \, d_\epsilon \, dxdt + \int_0^T \varphi(t) \int_\Omega A_\epsilon^{ijkl} \frac{1}{\epsilon} \omega_{kl}^\epsilon(\mathbf{u}_\epsilon(t)) \Upsilon_{ij}(\widehat{\mathbf{v}}_\epsilon) d_\epsilon \, dxdt +$$
$$+ \int_0^T \varphi(t) \int_\Omega A_\epsilon^{ijkl} \frac{1}{\epsilon} \omega_{kl}^\epsilon(\mathbf{u}_\epsilon(t)) \frac{1}{\epsilon} B_\epsilon^{ij} d_\epsilon \, dxdt = \int_0^T \varphi(t) \int_\Omega \mathbf{f}(t) \cdot \widehat{\mathbf{v}}_\epsilon \, d_\epsilon \, dxdt +$$
$$+ \int_0^T \varphi(t) \int_0^l \int_{\partial S} \mathbf{h}(t) \cdot \widehat{\mathbf{v}}_\epsilon \, d_\epsilon \epsilon \sqrt{\nu_j o^{ij,\epsilon} \nu_j} \, dS dx_1 dt. \tag{5.1.119}$$

Relations (5.1.19), (5.1.87), and (5.1.88) enable us to rewrite equation (5.1.119) as

$$\int_\Omega \rho \partial_t \mathbf{u}_\epsilon(t) \cdot \widehat{\mathbf{v}}_\epsilon \, d_\epsilon \, dx - \int_\Omega \rho \mathbf{q}_{1,\epsilon} \cdot \widehat{\mathbf{v}}_\epsilon \, d_\epsilon \, dx =$$
$$= -\int_0^t \int_\Omega A_\epsilon^{ijkl} \frac{1}{\epsilon} \omega_{kl}^\epsilon(\mathbf{u}_\epsilon(s)) \Upsilon_{ij}(\widehat{\mathbf{v}}_\epsilon) d_\epsilon \, dxds - \int_0^t \int_\Omega A_\epsilon^{ijkl} \frac{1}{\epsilon} \omega_{kl}^\epsilon(\mathbf{u}_\epsilon(s)) \frac{1}{\epsilon} B_\epsilon^{ij} d_\epsilon \, dxdt +$$
$$+ \int_0^t \int_\Omega \mathbf{f}(s) \cdot \widehat{\mathbf{v}}_\epsilon \, d_\epsilon \, dxds + \int_0^t \int_0^l \int_{\partial S} \mathbf{h}(s) \cdot \widehat{\mathbf{v}}_\epsilon \, d_\epsilon \epsilon \sqrt{\nu_j o^{ij,\epsilon} \nu_j} \, dS dx_1 ds. \tag{5.1.120}$$

The right-hand side of the equation (5.1.120) is convergent. We showed in the proof of Lemma 5.1.14 that

$$\widehat{\mathbf{v}}_\epsilon \to \mathbf{v} \text{ in } W^{1,2}(\Omega)^3$$

for $\epsilon \to 0$. Hence and from (5.1.17) and (5.1.90), we get that

$$\int_\Omega \rho(\partial_t \mathbf{u}_\epsilon(t) - \mathbf{q}_{1,\epsilon}) \cdot \widehat{\mathbf{v}}_\epsilon \, d_\epsilon \, dx \to \int_0^l \check{\rho}(\partial_t \mathbf{u}(t) - \mathbf{q}_1) \cdot \mathbf{v} \, dx_1 \text{ in } C([0,T]).$$

The rest of the proof is obvious. □

We have proved that the asymptotic dynamic model for the curved rod has the form:

$$-\int_0^T \dot{\varphi}(t) \int_0^l \check{\rho} \partial_t \mathbf{u}(t) \cdot \mathbf{v} \, dx_1 dt + \int_0^T \varphi(t) \int_0^l E[I_1(\partial_1 \mathbf{u}_*(t) \cdot \mathbf{b})(\mathbf{v}_*' \cdot \mathbf{b}) +$$
$$+ I_2(\partial_1 \mathbf{u}_*(t) \cdot \mathbf{n})(\mathbf{v}_*' \cdot \mathbf{n})] \, dx_1 dt + \int_0^T \varphi(t) \int_0^l \mu K(\partial_1 \mathbf{u}_*(t) \cdot \mathbf{t})(\mathbf{v}_*' \cdot \mathbf{t}) \, dx_1 dt =$$
$$= \int_0^T \varphi(t) \int_0^l \check{\mathbf{f}}_{\mathbf{f}+\mathbf{h}}(t) \cdot \mathbf{v} \, dx_1 dt \tag{5.1.121}$$

for all functions $\varphi \in C_0^\infty([0,T])$ and $\mathbf{v}_* \in W_0^{1,2}(0,l)^3$ generated by couples $\langle \mathbf{v}, \psi \rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$. The function $\mathbf{u}$, that together with the function ϕ generate the function $\mathbf{u}_*$, satisfies the initial state

$$\mathbf{u}|_{t=0} = \mathbf{q}_0 \text{ and } \check{\rho}\partial_t \mathbf{u}|_{t=0} = \check{\rho}\mathbf{q}_1 \tag{5.1.122}$$

in the sense of the space $C([0,T]; L^2(0,l)^3)$ and $C([0,T]; [\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)]')$, respectively.

Now, we decide upon the uniqueness of the solution to (5.1.121)–(5.1.122).

Lemma 5.1.17 *There exists the unique solution to the equation* (5.1.121) *satisfying* (5.1.122).

Proof: Suppose that there exist two solutions $\langle \mathbf{u}_j, \phi_j\rangle \in L^\infty(0,T;\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l))$ such that $\partial_t \mathbf{u}_j \in L^\infty(0,T;L^2(0,l)^3)\cap C([0,T];[\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)]')$, $j=1,2$. Let us denote $\widehat{\mathbf{u}} = \mathbf{u}_1 - \mathbf{u}_2$ and $\widehat{\phi} = \phi_1 - \phi_2$. Then the couple $\langle \widehat{\mathbf{u}}, \widehat{\phi}\rangle \in L^\infty(0,T;\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l))$, $\widehat{\mathbf{u}}_* \in L^\infty(0,T;W_0^{1,2}(0,l)^3)$, $\partial_t\widehat{\mathbf{u}} \in L^\infty(0,T;L^2(0,l)^3)\cap C([0,T];[\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)]')$, and satisfies

$$-\int_0^T \dot{\varphi}(t)\int_0^l \check{\rho}\partial_t\widehat{\mathbf{u}}(t)\cdot\mathbf{v}\;dx_1dt + \int_0^T \varphi(t)\int_0^l E[I_1(\partial_1\widehat{\mathbf{u}}_*(t)\cdot\mathbf{b})(\mathbf{v}'_*\cdot\mathbf{b})+$$

$$+I_2(\partial_1\widehat{\mathbf{u}}_*(t)\cdot\mathbf{n})(\mathbf{v}'_*\cdot\mathbf{n})] + \mu K(\partial_1\widehat{\mathbf{u}}_*(t)\cdot\mathbf{t})(\mathbf{v}'_*\cdot\mathbf{t})\;dx_1dt = 0 \qquad (5.1.123)$$

and the initial state

$$\widehat{\mathbf{u}}|_{t=0} = 0 \text{ and } \check{\rho}\partial_t\widehat{\mathbf{u}}|_{t=0} = 0 \qquad (5.1.124)$$

in the sense of the space $C([0,T];L^2(0,l)^3)$ and $C([0,T];[\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)]')$, respectively.

We can rewrite again (5.1.123) as

$$\int_0^l \check{\rho}\partial_t\widehat{\mathbf{u}}(t)\cdot\mathbf{v}\;dx_1 + \int_0^t\int_0^l E[I_1(\partial_1\widehat{\mathbf{u}}_*(s)\cdot\mathbf{b})(\mathbf{v}'_*\cdot\mathbf{b}) + I_2(\partial_1\widehat{\mathbf{u}}_*(s)\cdot\mathbf{n})(\mathbf{v}'_*\cdot\mathbf{n})]+$$

$$+\mu K(\partial_1\widehat{\mathbf{u}}_*(s)\cdot\mathbf{t})(\mathbf{v}'_*\cdot\mathbf{t})\;dx_1ds = 0 \qquad (5.1.125)$$

for all $t\in[0,T]$. Since $\langle\widehat{\mathbf{u}}(t),\widehat{\phi}(t)\rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ for a.a. $t\in(0,T)$, we can use this couple as a test function in (5.1.125), and we get that

$$\int_0^l \check{\rho}\partial_t\widehat{\mathbf{u}}(t)\cdot\widehat{\mathbf{u}}(t)\;dx_1 + \int_0^t\int_0^l E[I_1(\partial_1\widehat{\mathbf{u}}_*(s)\cdot\mathbf{b})(\partial_1\widehat{\mathbf{u}}_*(t)\cdot\mathbf{b})+$$

$$+I_2(\partial_1\widehat{\mathbf{u}}_*(s)\cdot\mathbf{n})(\partial_1\widehat{\mathbf{u}}_*(t)\cdot\mathbf{n})] + \mu K(\partial_1\widehat{\mathbf{u}}_*(s)\cdot\mathbf{t})(\partial_1\widehat{\mathbf{u}}_*(t)\cdot\mathbf{t})\;dx_1ds = 0 \quad (5.1.126)$$

for all $t\in[0,T]$. It is obvious that equation (5.1.126) can be rewritten as

$$\frac{d}{dt}\int_0^l \frac{\check{\rho}|\widehat{\mathbf{u}}(t)|^2}{2}\;dx_1 + \frac{d}{dt}\int_0^l \frac{EI_1}{2}\left(\int_0^t \partial_1\widehat{\mathbf{u}}_*(s)\cdot\mathbf{b}\;ds\right)^2\;dx_1+$$

$$+\frac{d}{dt}\int_0^l \frac{EI_2}{2}\left(\int_0^t \partial_1\widehat{\mathbf{u}}_*(s)\cdot\mathbf{n}\;ds\right)^2\;dx_1+$$

$$+\frac{d}{dt}\int_0^l \frac{\mu K}{2}\left(\int_0^t \partial_1\widehat{\mathbf{u}}_*(s)\cdot\mathbf{t}\;ds\right)^2\;dx_1 = 0 \qquad (5.1.127)$$

for all $t\in[0,T]$. It follows from the assumptions on the functions $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{u}}_*$ that the functions $\widehat{\mathbf{u}}$ and $\int_0^{\cdot}\partial_1\widehat{\mathbf{u}} \in C([0,T];L^2(0,l)^3)$, which enables us to integrate (5.1.127) over the interval $[0,t]$, and we get from (5.1.124) that

$$\int_0^l \frac{\check{\rho}|\widehat{\mathbf{u}}(t)|^2}{2}\;dx_1 + \int_0^l \frac{EI_1}{2}\left(\int_0^t \partial_1\widehat{\mathbf{u}}_*(s)\cdot\mathbf{b}\;ds\right)^2\;dx_1+$$

$$+\int_0^l \frac{EI_2}{2}\left(\int_0^t \partial_1\widehat{\mathbf{u}}_*(s)\cdot\mathbf{n}\;ds\right)^2\;dx_1+$$

$$+\int_0^l \frac{\mu K}{2}\left(\int_0^t \partial_1\widehat{\mathbf{u}}_*(s)\cdot\mathbf{t}\;ds\right)^2\;dx_1 = 0 \qquad (5.1.128)$$

for all $t \in [0,T]$. Hence $\widehat{\mathbf{u}} \equiv \mathbf{0}$ as a consequence of (5.1.15) and non-negativity of all terms in (5.1.128). Further, (5.1.128) yields that

$$\int_0^t \partial_1 \widehat{\mathbf{u}}_*(x_1,s)\ ds = \int_0^t [(\partial_1 \widehat{\mathbf{u}}_*(x_1,s)\cdot \mathbf{t}(x_1))\mathbf{t}(x_1)+$$

$$+(\partial_1 \widehat{\mathbf{u}}_*(x_1,s)\cdot \mathbf{n}(x_1))\mathbf{n}(x_1) + (\partial_1 \widehat{\mathbf{u}}_*(x_1,s)\cdot \mathbf{b}(x_1))\mathbf{b}(x_1)]\ ds = 0$$

for all $t \in [0,T]$ and for arbitrary but fixed $x_1 \in (0,l)$. Then $\partial_1 \widehat{\mathbf{u}}_*(x_1,t) = 0$ for a.a $(x_1,t) \in (0,l)\times(0,T)$. Since $\widehat{\mathbf{u}}_* \in W_0^{1,2}(0,l)$ then also $\widehat{\mathbf{u}}_* \equiv \mathbf{0}$ and $\phi = -\widehat{\mathbf{u}}_* \cdot \mathbf{t} = 0$, a contradiction. □

As a consequence of the uniqueness, we can claim that it is not necessary to pass to subsequences in (5.1.89)–(5.1.91).

At the end, we go back to the original curve $\mathcal{C}$ described by the parametrization $\mathbf{\Phi}$. We introduce the following notation: $\widetilde{v} : \mathcal{C} \to \mathbb{R}$ and $\widetilde{v}(\Phi_1(x_1), \Phi_2(x_1), \Phi_3(x_1)) = v(x_1)$ for a.a. $x_1 \in (0,l)$. Then we can easily see that

$$v'(x_1) = [(\widetilde{\partial}_i \widetilde{v}) \circ \mathbf{\Phi}] t_i$$

and thus

$$\left[\frac{d}{dx_1}(\widetilde{v}\circ \mathbf{\Phi})\right] \circ \mathbf{\Phi}^{-1} = (\widetilde{\mathbf{t}} \cdot \widetilde{\nabla})\widetilde{v} = \frac{\partial \widetilde{v}}{\partial \widetilde{\mathbf{t}}}.$$

It enables us to rewrite the limit model (5.1.121) as follows

$$-\int_0^T \dot{\varphi}(t) \int_{\mathcal{C}} \widetilde{\rho}\partial_t \widetilde{\mathbf{u}}(t)\cdot \widetilde{\mathbf{v}}\ d\mathcal{C}dt + \int_0^T \varphi(t)\int_{\mathcal{C}} E\left[I_1\left(\frac{\partial \widetilde{\mathbf{u}}_*(t)}{\partial \widetilde{\mathbf{t}}}\cdot \widetilde{\mathbf{b}}\right)\left(\frac{\partial \widetilde{\mathbf{v}}_*}{\partial \widetilde{\mathbf{t}}}\cdot \widetilde{\mathbf{b}}\right) + \right.$$

$$\left. + I_2\left(\frac{\partial \widetilde{\mathbf{u}}_*(t)}{\partial \widetilde{\mathbf{t}}}\cdot \widetilde{\mathbf{n}}\right)\left(\frac{\partial \widetilde{\mathbf{v}}_*}{\partial \widetilde{\mathbf{t}}}\cdot \widetilde{\mathbf{n}}\right)\right]\ d\mathcal{C}dt+$$

$$+\int_0^T \varphi(t)\int_{\mathcal{C}} \mu K\left(\frac{\partial \widetilde{\mathbf{u}}_*(t)}{\partial \widetilde{\mathbf{t}}}\cdot \widetilde{\mathbf{t}}\right)\left(\frac{\partial \widetilde{\mathbf{v}}_*}{\partial \widetilde{\mathbf{t}}}\cdot \widetilde{\mathbf{t}}\right)\ d\mathcal{C}dt =$$

$$= \int_0^T \varphi(t)\int_{\mathcal{C}} \widetilde{\widetilde{\mathbf{f}}}_{\widetilde{\mathbf{f}}+\widetilde{\mathbf{h}}}(t)\cdot \widetilde{\mathbf{v}}\ d\mathcal{C}dt,\ \forall \widetilde{\mathbf{v}} \in W_0^{1,2}(\mathcal{C}). \tag{5.1.129}$$

5.2 Asymptotic analysis of heat conducting elastic materials

In this section, we continue with the study of the asymptotic behavior of elastic materials. We, however, assume now a thermodynamically consistent system. The more general version of the system was derived in [178]. We also refer the reader to Section 2 for more details. With regard to the reader, we repeat basic equations. The most general model from [178] consists of the equations

$$\rho \partial_{tt}\mathbf{u} - \operatorname{div}\sigma = \mathbf{f}, \tag{5.2.1}$$

$$\sigma := \operatorname{div}(\lambda \mathbf{u} + \lambda_v \partial_t \mathbf{u})\mathbb{I} + 2(\mu D\mathbf{u} + \mu_v D\partial_t \mathbf{u}) - \upsilon(3\lambda+2\mu)\vartheta\mathbb{I} + \gamma \operatorname{div}\nabla^2 \mathbf{u}, \tag{5.2.2}$$

$$c\partial_t\vartheta + \vartheta\partial_t(\upsilon(3\lambda+2\mu)\operatorname{div}\mathbf{u}) = \operatorname{div}(\kappa(\vartheta)\nabla\vartheta) + \lambda_v(\operatorname{div}\partial_t\mathbf{u})^2 + 2\mu_v|D\partial_t\mathbf{u}|^2 + h, \tag{5.2.3}$$

where

- $\mathbf{u} : Q\times(0,T) \to \mathbb{R}^3$ is a displacement,
- $\vartheta : Q\times(0,T) \to \mathbb{R}$ is temperature,
- D stands for the symmetric part of the gradient,

- $\lambda \geq 0$ and $\mu > 0$ are Lamé constants related to elastic response,
- υ is the coefficient of thermal expansion,
- $\gamma > 0$ is a regularizing coefficient reflecting bending rigidity,
- $c > 0$ is heat capacity,
- $\kappa > 0$ is heat conduction function,
- $\rho > 0$ is mass density,
- $\lambda_v \geq 0$ and $\mu_v > 0$ are Lamé constants related to viscous response,
- $\mathbf{f} : Q \times (0,T) \to \mathbb{R}^3$ is an external force,
- $h : Q \times (0,T) \to \mathbb{R}$ is an internal heat source.

The introduced system is, however, very complicated. We thus make additional assumptions that enable us to reduce its complexity. We thus assume that the displacements and their velocities are small and the higher order terms in (5.2.1)–(5.2.3) can be neglected and the system is then governed by the equations (3.0.5)–(3.0.6).

Existence of a solution to (3.0.5)–(3.0.9) was proved in [215]. Unfortunately, the existence is only local in time under suitable assumptions on the initial and boundary conditions. Despite the deficiency, we can still apply our dimension reduction approach that is not related to the existence of a global-in-time solution. In the next section, a weak formulation of the system is introduced and the main results are stated. Then we apply the main ideas of our dimension reduction approach from the previous section and we study the asymptotic behavior of the system on thin curved domains.

5.2.1 Preliminaries

First, we introduce the existence result under the boundary conditions (3.0.7)–(3.0.8), where the two-dimensional measures of Γ_1 and Γ_2 are not equal to zero and $\mathbf{n}$ is the unit outward normal to ∂Q. We further denote

$$W(Q) := \{\mathbf{v} \in W^{1,2}(Q)^3 : \ \mathbf{v}|_{\Gamma_1} = 0\}. \tag{5.2.4}$$

The weak formulation of (3.0.5)–(3.0.6) under conditions (3.0.7)–(3.0.9) reads

$$-\int_0^T \int_Q \partial_t \mathbf{u} \cdot \partial_t \mathbf{v} \; dxdt + \int_Q \mathbf{u}_1(x) \cdot \mathbf{v}(x,0) \; dx + \int_0^T \int_Q A^{ijkl} D_{kl}\mathbf{u} D_{ij}\mathbf{v} \; dxdt +$$

$$+ \int_0^T \int_Q B^{ijkl} D_{kl}\partial_t \mathbf{u} D_{ij}\mathbf{v} \; dxdt + \int_0^T \int_Q \nabla\vartheta \cdot \mathbf{v} \; dxdt = \int_0^T \int_Q \mathbf{f} \cdot \mathbf{v} \; dxdt \tag{5.2.5}$$

for any $\mathbf{v} \in C^1([0,T]; W(Q))$, where $\mathbf{v}(x,T) = \mathbf{0}$, and

$$A^{ijkl} := \lambda \delta^{ij}\delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}), \;\; B^{ijkl} := \lambda_v \delta^{ij}\delta^{kl} + \mu_v(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}),$$

and

$$-\int_0^T \int_Q \vartheta \partial_t \psi \; dxdt + \int_Q \vartheta_0(x)\psi(x,0) \; dx +$$

$$+ \int_0^T \int_Q \kappa(\vartheta)\nabla\vartheta \cdot \nabla\psi \; dxdt + \int_0^T \int_Q \vartheta\psi \operatorname{div} \partial_t \mathbf{u} \; dxdt = \int_0^T \int_Q h\psi \; dxdt \tag{5.2.6}$$

for all $\psi \in C^1([0,T];C^1(\overline{Q}))$, $\psi(x,T) = 0$. Note that we put $\rho = c = 1$ and $\upsilon = \frac{1}{3\lambda+2\mu}$. This choice of the constants affects only the time interval where we can prove the existence of a solution.

The main existence result states:

Theorem 5.2.1 [215] *Let $\mathbf{u}_0$, $\mathbf{u}_1$, $\vartheta_0 \in L^2(Q)$, $\vartheta_0 \geq 0$, $\mathbf{f} \in L^2(0,T;L^2(Q)^3)$, $h \in L^2(0,T;L^2(Q))$, $h \geq 0$, and let $\kappa(z)$ be a nondecreasing function such that*

$$C_1(z^{\alpha-1}+1) \leq \kappa(z) \leq C_2(z^{\alpha-1}+1),\ (\kappa^{\frac{1}{2}}(z))' \leq C_3 z^{\frac{\alpha-3}{2}} + C_4 \tag{5.2.7}$$

for $C_i > 0$, $i = 1,\ldots,4$, $\alpha > 3$, and $z \geq 0$. Then there exists couple $\langle \mathbf{u}, \vartheta\rangle$ solving (5.2.5)–(5.2.6) *in time interval $(0,T)$, where T depends on $\mathbf{u}_0$, $\mathbf{u}_1$, ϑ_0, such that*

$$\mathbf{u} \in C([0,T];W(Q)),\ \partial_t \mathbf{u} \in L^2(0,T;W(Q)) \cap C([0,T];[W(Q)]'), \tag{5.2.8}$$

$$\vartheta \in L^2(0,T;W^{1,2}(Q)) \cap C([0,T];[W^{1,2}(Q)]'). \tag{5.2.9}$$

In addition, the solution satisfies the energy inequality

$$\frac{1}{2}\int_Q [|\partial_t \mathbf{u}(t)|^2 + A^{ijkl}D_{kl}\mathbf{u}(t)D_{ij}\mathbf{u}(t) + \vartheta^2(t)]\ dx + \int_0^t \int_Q [B^{ijkl}D_{kl}\partial_t\mathbf{u}(s)D_{ij}\partial_t\mathbf{u}(s) +$$

$$+|\nabla K_{\frac{1}{2}}(\vartheta(s))|^2]\ dxds \leq \frac{1}{2}\int_Q [|\mathbf{u}_1|^2 + A^{ijkl}D_{kl}\mathbf{u}_0 D_{ij}\mathbf{u}_0 + \vartheta_0^2]\ dx +$$

$$+\int_0^t \int_Q [\mathbf{f}(s)\cdot\partial_t\mathbf{u}(s) + h(s)\vartheta(s)]\ dxds -$$

$$-\int_0^t\int_Q \nabla\vartheta(s)\cdot\partial_t\mathbf{u}(s)\ dxds - \int_0^t\int_Q \vartheta^2(s)\operatorname{div}\partial_t\mathbf{u}(s)\ dxds \tag{5.2.10}$$

for a.a. $t \in (0,T)$, where $K_{\frac{1}{2}}(z) = \int_0^z \kappa^{\frac{1}{2}}(y)\ dy$.

Let us now assume that Q is a thin domain. We can use the definitions (4.2.2) and (4.2.4) for mappings $\mathbf{R}_\epsilon$ and $\bar{\mathbf{P}}_\epsilon$, respectively, and put again $\widetilde{\Omega}_\epsilon := (\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)(\Omega)$, where $\Omega := (0,l)\times S$. The weak formulation (3.0.5)–(3.0.6) for $Q = \widetilde{\Omega}_\epsilon$ is

$$-\int_0^T\int_{\widetilde{\Omega}_\epsilon} \partial_t\widetilde{\mathbf{u}}_\epsilon \cdot \partial_t\widetilde{\mathbf{v}}_\epsilon\ dxdt + \int_{\widetilde{\Omega}_\epsilon} \widetilde{\mathbf{u}}_{1,\epsilon}\cdot\widetilde{\mathbf{v}}_\epsilon(\cdot,0)\ d\widetilde{y} +$$

$$+\int_0^T\int_{\widetilde{\Omega}_\epsilon} (\widetilde{B}_\epsilon^{ijkl}\widetilde{D}_{kl}\partial_t\widetilde{\mathbf{u}}_\epsilon + \widetilde{A}_\epsilon^{ijkl}\widetilde{D}_{kl}\widetilde{\mathbf{u}}_\epsilon)\widetilde{D}_{ij}\widetilde{\mathbf{v}}_\epsilon\ d\widetilde{y}dt =$$

$$= \int_0^T\int_{\widetilde{\Omega}_\epsilon} \widetilde{\mathbf{f}}_\epsilon\cdot\widetilde{\mathbf{v}}_\epsilon\ d\widetilde{y}dt - \upsilon_\epsilon(3\lambda_\epsilon + 2\mu_\epsilon)\int_0^T\int_{\widetilde{\Omega}_\epsilon} \widetilde{\nabla}\widetilde{\vartheta}_\epsilon\cdot\widetilde{\mathbf{v}}_\epsilon\ d\widetilde{y}dt, \tag{5.2.11}$$

where $\widetilde{A}_\epsilon^{ijkl} := \lambda_\epsilon\delta^{ij}\delta^{kl} + \mu_\epsilon(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})$ and $\widetilde{B}_\epsilon^{ijkl} := \lambda_{\upsilon,\epsilon}\delta^{ij}\delta^{kl} + \mu_{\upsilon,\epsilon}(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})$,

$$-\int_0^T\int_{\widetilde{\Omega}_\epsilon} \widetilde{c}_\epsilon\widetilde{\vartheta}_\epsilon\partial_t\widetilde{\psi}_\epsilon\ d\widetilde{y}dt + \int_{\widetilde{\Omega}_\epsilon} \widetilde{c}_\epsilon\widetilde{\vartheta}_{0,\epsilon}\widetilde{\psi}_\epsilon(\cdot,0)\ d\widetilde{y} + \int_0^T\int_{\widetilde{\Omega}_\epsilon} \widetilde{\kappa}_\epsilon(\widetilde{\vartheta}_\epsilon)\widetilde{\nabla}\widetilde{\vartheta}_\epsilon\cdot\widetilde{\nabla}\widetilde{\psi}_\epsilon\ d\widetilde{y}dt +$$

$$+\upsilon_\epsilon(3\lambda_\epsilon + 2\mu_\epsilon)\int_0^T\int_{\widetilde{\Omega}_\epsilon} \widetilde{\vartheta}_\epsilon\widetilde{\operatorname{div}}\,\partial_t\widetilde{\mathbf{u}}_\epsilon\widetilde{\psi}_\epsilon\ d\widetilde{y}dt = \int_0^T\int_{\widetilde{\Omega}_\epsilon} \widetilde{h}_\epsilon\widetilde{\psi}_\epsilon\ d\widetilde{y}dt. \tag{5.2.12}$$

Using the scaling

$$\lambda_{\upsilon,\epsilon} := \frac{\lambda_\upsilon}{\epsilon^2},\ \mu_{\upsilon,\epsilon} := \frac{\mu_\upsilon}{\epsilon^2},\ \lambda_\epsilon := \frac{\lambda}{\epsilon^2},\ \mu_\epsilon := \frac{\mu}{\epsilon^2},\ \upsilon_\epsilon := \epsilon^2\upsilon, \tag{5.2.13}$$

$$\widetilde{c}_\epsilon := \epsilon,\ \widetilde{\kappa}_\epsilon(z) := \epsilon\widetilde{\kappa}(z),\ \widetilde{h}_\epsilon(\widetilde{y},t) := \epsilon\widetilde{h}(\widetilde{y},t), \tag{5.2.14}$$

and the corresponding notation from Section 2, 3 and 4 together with (5.1.5)–(5.1.8), we can pass from (5.2.11)–(5.2.12) to the weak formulation on referential domain Ω

$$-\int_0^T\int_\Omega \partial_t\mathbf{u}_\epsilon\cdot\partial_t\mathbf{v}d_\epsilon\ dxdt + \int_\Omega \mathbf{u}_{1,\epsilon}\cdot\mathbf{v}(\cdot,0)d_\epsilon\ dx+$$

$$+\frac{1}{\epsilon^2}\int_0^T\int_\Omega B_\epsilon^{ijkl}\omega_{kl}^\epsilon(\partial_t\mathbf{u}_\epsilon)\omega_{ij}^\epsilon(\mathbf{v})d_\epsilon\ dxdt + \frac{1}{\epsilon^2}\int_0^T\int_\Omega A_\epsilon^{ijkl}\omega_{kl}^\epsilon(\mathbf{u}_\epsilon)\omega_{ij}^\epsilon(\mathbf{v})d_\epsilon\ dxdt =$$

$$= \int_0^T\int_\Omega \mathbf{f}\cdot\mathbf{v}d_\epsilon\ dxdt - \upsilon(3\lambda+2\mu)\int_0^T\int_\Omega \partial_k^\epsilon\vartheta_\epsilon(\mathbf{g}^{k,\epsilon}\cdot\mathbf{v})d_\epsilon\ dxdt \tag{5.2.15}$$

for any $\mathbf{v}\in C^1([0,T];V(\Omega))$, $\mathbf{v}(\cdot,T)=0$, where $A_\epsilon^{ijkl} = \lambda g^{ij,\epsilon}g^{kl,\epsilon} + \mu(g^{ik,\epsilon}g^{jl,\epsilon} + g^{il,\epsilon}g^{jk,\epsilon})$ and $B_\epsilon^{ijkl} = \lambda_v g^{ij,\epsilon}g^{kl,\epsilon} + \mu_v(g^{ik,\epsilon}g^{jl,\epsilon} + g^{il,\epsilon}g^{jk,\epsilon})$, and

$$-\int_0^T\int_\Omega \vartheta_\epsilon\partial_t\psi d_\epsilon\ dxdt + \int_\Omega \vartheta_{0,\epsilon}\psi(\cdot,0)d_\epsilon\ dx + \int_0^T\int_\Omega \kappa(\vartheta_\epsilon)g_\epsilon^{kl}\partial_k^\epsilon\vartheta_\epsilon\partial_l^\epsilon\psi d_\epsilon\ dxdt+$$

$$+\frac{\upsilon(3\lambda+2\mu)}{\epsilon}\int_0^T\int_\Omega g_\epsilon^{kl}\vartheta_\epsilon\omega_{kl}^\epsilon(\partial_t\mathbf{u}_\epsilon)\psi d_\epsilon\ dxdt = \int_0^T\int_\Omega h\psi d_\epsilon\ dxdt \tag{5.2.16}$$

for any $\psi\in C^1([0,T];C^1(\overline{\Omega}))$ such that $\psi(\cdot,T)=0$. Here we use notation $\nabla^\epsilon = (\partial_1^\epsilon,\partial_2^\epsilon,\partial_3^\epsilon) := (\partial_1, \frac{\partial_2}{\epsilon}, \frac{\partial_3}{\epsilon})$. The corresponding energy inequality reads

$$\frac{1}{2}\int_\Omega[|\partial_t\mathbf{u}_\epsilon(t)|^2 + \vartheta_\epsilon^2(t)]d_\epsilon\ dx + \frac{1}{2\epsilon^2}\int_\Omega A_\epsilon^{ijkl}\omega_{kl}^\epsilon(\mathbf{u}_\epsilon(t))\omega_{ij}^\epsilon(\mathbf{u}_\epsilon(t))d_\epsilon\ dx+$$

$$+\frac{1}{\epsilon^2}\int_0^t\int_\Omega B_\epsilon^{ijkl}\omega_{kl}^\epsilon(\partial_t\mathbf{u}_\epsilon)\omega_{ij}^\epsilon(\partial_t\mathbf{u}_\epsilon)d_\epsilon\ dxds + \int_0^t\int_\Omega g_\epsilon^{kl}\partial_k^\epsilon K_{\frac12}(\vartheta_\epsilon)\partial_l^\epsilon K_{\frac12}(\vartheta_\epsilon)d_\epsilon\ dxds \le$$

$$\le \frac{1}{2}\int_\Omega[|\mathbf{u}_{1,\epsilon}|^2 + \vartheta_{0,\epsilon}^2]d_\epsilon\ dx + \frac{1}{2\epsilon^2}\int_\Omega A_\epsilon^{ijkl}\omega_{kl}^\epsilon(\mathbf{u}_{0,\epsilon})\omega_{ij}^\epsilon(\mathbf{u}_{0,\epsilon})d_\epsilon\ dx+$$

$$+\int_0^t\int_\Omega[\mathbf{f}\cdot\partial_t\mathbf{u}_\epsilon + h\vartheta_\epsilon]d_\epsilon\ dxds-$$

$$-\upsilon(3\lambda+2\mu)\int_0^t\int_\Omega[\partial_k^\epsilon\vartheta_\epsilon(\mathbf{g}^{k,\epsilon}\cdot\partial_t\mathbf{u}_\epsilon) - \frac{g_\epsilon^{kl}}{\epsilon}\vartheta_\epsilon^2\omega_{kl}^\epsilon(\partial_t\mathbf{u}_\epsilon)]d_\epsilon\ dxds. \tag{5.2.17}$$

Relation (5.2.17) is a consequence of the fact that (5.2.10) arises as a sum of two identities in the proof of the existence of a solution and following limit passages. The first one of the identities comes from (5.2.5) and the second one from a suitable approximation of (5.2.6), which enables us to change constants depending on ϵ in such way that leads to (5.2.17).

Let us denote that constants I_1, I_2 and K were defined in (5.1.105)–(5.1.106). The next constants we need are

$$F := \frac{\lambda+\mu}{\lambda_v+\mu_v},\ G := \frac{3\lambda+2\mu}{3\lambda_v+2\mu_v}. \tag{5.2.18}$$

The main result regarding the asymptotic analysis states:

Theorem 5.2.2 *We assume that function $\mathbf{\Phi}\in W^{1,\infty}(0,l)^3$ is a parametrization of a Jordan unit speed curve generating a local frame $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}\in L^\infty(0,l)^3$. Let functions $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$ satisfying* (4.2.3), (4.3.2)–(4.3.9) *be smooth approximations of this local frame. Let, further, $\mathbf{f}\in L^2(0,T;L^2(\Omega)^3)$, $h\in L^2(0,T;L^2(\Omega))$, $h\ge 0$, and κ satisfies* (5.2.7)*. We assume that there are no constants C_i, $i=1,2,3$, such that*

$C_1t_1(x_1) = C_2t_2(x_1) = C_3t_3(x_1)$ for almost all $x_1 \in (0,l)$ and $\mathbf{t} = (t_1,t_2,t_3)$, and let

$$\frac{1}{\epsilon}\omega^{\epsilon}_{ii}(\mathbf{u}_{0,\epsilon}) \rightharpoonup \zeta_{\mathbf{u}_0,ii} \text{ in } L^2(\Omega). \tag{5.2.19}$$

We define

$$\begin{aligned}
w_1(x_1,t) &:= (\lambda-\lambda_v F)e^{-Ft}\int_S(\zeta_{\mathbf{u}_0,11}+\zeta_{\mathbf{u}_0,22}+\zeta_{\mathbf{u}_0,33})x_2\ dx_2dx_3 + \\
&\quad +\frac{\lambda-\lambda_v F}{\lambda_v+\mu_v}e^{-Ft}\int_0^t e^{Fs}I_1[\mu_v(\partial_t\partial_1\mathbf{u}_*(s)\cdot\mathbf{b})+\mu(\partial_1\mathbf{u}_*(s)\cdot\mathbf{b})]\ ds,\\
w_2(x_1,t) &:= (\lambda_v F-\lambda)e^{-Ft}\int_S(\zeta_{\mathbf{u}_0,11}+\zeta_{\mathbf{u}_0,22}+\zeta_{\mathbf{u}_0,33})x_3\ dx_2dx_3 + \\
&\quad +\frac{\lambda-\lambda_v F}{\lambda_v+\mu_v}e^{-Ft}\int_0^t e^{Fs}I_2[\mu_v(\partial_t\partial_1\mathbf{u}_*(s)\cdot\mathbf{n})+\mu(\partial_1\mathbf{u}_*(s)\cdot\mathbf{n})]\ ds,\\
w_3(x_1,t) &:= -Ge^{-Gt}\int_S\zeta_{\mathbf{u}_0,ii}\ dx_2dx_3.
\end{aligned}$$

We further assume

$$\mathbf{u}_{0,\epsilon}\to\mathbf{u}_0=\mathbf{u}_0(x_1),\ \ \mathbf{u}_{1,\epsilon}\to\mathbf{u}_1=\mathbf{u}_1(x_1) \text{ in } L^2(\Omega)^3 \tag{5.2.20}$$

and

$$\vartheta_{0,\epsilon}\to\vartheta_0=\vartheta_0(x_1) \text{ in } L^2(\Omega). \tag{5.2.21}$$

Then there exists pair $\langle\mathbf{u},\phi_{\mathbf{u}}\rangle \in L^\infty(0,T;\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l))$, $T=T(\mathbf{u}_0,\mathbf{u}_1,\vartheta_0)$, generating $\mathbf{u}_*$ (see (4.2.17)) and such that $\partial_t\mathbf{u}\in L^\infty(0,T;L^2(0,l)^3)$. $\langle\mathbf{u},\phi_{\mathbf{u}}\rangle$ solves the equations

$$\begin{aligned}
&-|S|\int_0^T\int_0^l\partial_t\mathbf{u}\cdot\partial_t\mathbf{v}\ dx_1dt+|S|\int_0^l\mathbf{u}_1\cdot\mathbf{v}(\cdot,0)\ dx_1+\\
&+\int_0^T\int_0^l\Big[\mu_vI_1\left(2+\frac{\lambda_v}{\lambda_v+\mu_v}\right)\partial_t\partial_1\mathbf{u}_*\cdot\mathbf{b}+\\
&+\mu I_1\left(2+\frac{\lambda}{\lambda_v+\mu_v}\right)\partial_1\mathbf{u}_*\cdot\mathbf{b}+w_1\Big]\partial_1\mathbf{v}_*\cdot\mathbf{b}\ dx_1dt+\\
&+\int_0^T\int_0^l\Big[\mu_vI_2\left(2+\frac{\lambda_v}{\lambda_v+\mu_v}\right)\partial_t\partial_1\mathbf{u}_*\cdot\mathbf{n}+\\
&+\mu I_2\left(2+\frac{\lambda}{\lambda_v+\mu_v}\right)\partial_1\mathbf{u}_*\cdot\mathbf{n}+w_2\Big]\partial_1\mathbf{v}_*\cdot\mathbf{n}\ dx_1dt+\\
&+\int_0^T\int_0^l[\mu_vK(\partial_t\partial_1\mathbf{u}_*\cdot\mathbf{t})+\mu K(\partial_1\mathbf{u}_*\cdot\mathbf{t})]\partial_1\mathbf{v}_*\cdot\mathbf{t}\ dx_1dt=\\
&=\int_0^T\int_0^l\check{\mathbf{f}}\cdot\mathbf{v}\ dx_1dt-|S|\int_0^T\int_0^l\partial_1\vartheta\mathbf{v}\cdot\mathbf{t}\ dx_1dt
\end{aligned} \tag{5.2.22}$$

for all functions $\mathbf{v}_*\in C^1([0,T];W_0^{1,2}(0,l)^3)$ generated by the couples $\langle\mathbf{v},\phi_{\mathbf{v}}\rangle\in C^1([0,T];\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l))$, $\mathbf{v}(\cdot,T)=0$, where $\check{\mathbf{f}}=\int_S\mathbf{f}\ dx_2dx_3$, and

$$\begin{aligned}
&-\int_0^T\int_0^l\vartheta\partial_t\psi\ dx_1dt+\int_0^l\vartheta_0\psi(\cdot,0)\ dx_1+\int_0^T\int_0^l\kappa(\vartheta)\partial_1\vartheta\partial_1\psi\ dx_1dt+\\
&+\frac{\upsilon(3\lambda+2\mu)}{|S|}\int_0^T\int_0^l\vartheta w_3\psi\ dx_1dt=\int_0^T\int_0^l\check{h}\psi\ dx_1dt
\end{aligned} \tag{5.2.23}$$

for any $\psi \in C^1([0,T];C^1([0,l]))$, $\psi(\cdot,T)=0$, *where* $\check{h} := \frac{1}{|S|}\int_S h\ dx_2dx_3$. *In addition,*

$$\mathbf{u}_{\epsilon_n} \overset{*}{\rightharpoonup} \mathbf{u} \text{ in } L^\infty(0,T;V(\Omega)), \tag{5.2.24}$$

$$\frac{1}{2\epsilon_n}(\partial_2\mathbf{u}_{\epsilon_n}\cdot\mathbf{b}_{\epsilon_n} - \partial_3\mathbf{u}_{\epsilon_n}\cdot\mathbf{b}_{\epsilon_n}) \overset{*}{\rightharpoonup} \phi_{\mathbf{u}} \text{ in } L^\infty(0,T;L^2(\Omega)), \tag{5.2.25}$$

$$\vartheta_{\epsilon_n} \overset{*}{\rightharpoonup} \vartheta \text{ in } L^\infty(0,T;L^2(\Omega)) \tag{5.2.26}$$

for $\epsilon_n \to 0$, *where couples* $\langle \mathbf{u}_{\epsilon_n}, \vartheta_{\epsilon_n}\rangle$ *are solutions to* (5.2.15)–(5.2.16) *and satisfy* (5.2.17) *for each positive* ϵ_n *sufficiently small.*

5.2.2 Proof of Theorem 5.2.2

In this section, we present the proof of Theorem 5.2.2. First, let us recall the estimate (5.1.20) for A_ϵ^{ijkl}. A similar estimate can be derived for B_ϵ^{ijkl}. The next important ingredient is Korn's inequality (5.1.67). Instead of (5.1.73) we have

$$A_\epsilon^{ijkl} \to A_0^{ijkl},\ B_\epsilon^{ijkl} \to B_0^{ijkl} \text{ in } C(\overline{\Omega}), \tag{5.2.27}$$

where $A_0^{ijkl} = \lambda\delta^{ij}\delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})$ and $B_0^{ijkl} = \lambda_v\delta^{ij}\delta^{kl} + \mu_v(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})$.

To pass to the limit for $\epsilon \to 0$ we must find suitable estimates of the terms on the right-hand side of the energy inequality (5.2.17). By virtue of (5.1.25), the terms on the right-hand side of (5.2.17) can be estimated as follows

1.
$$\left|\int_0^t\int_\Omega \mathbf{f}\cdot\partial_t\mathbf{u}_\epsilon d_\epsilon\ dxds\right| \le \frac{1}{2}\int_0^t\int_\Omega |\mathbf{f}|^2 d_\epsilon\ dxds + \frac{1}{2}\int_0^t\int_\Omega |\partial_t\mathbf{u}_\epsilon|^2 d_\epsilon\ dxds,$$

2.
$$\left|\int_0^t\int_\Omega h\vartheta_\epsilon d_\epsilon\ dxds\right| \le \frac{1}{2}\int_0^t\int_\Omega h^2 d_\epsilon\ dxds + \frac{1}{2}\int_0^t\int_\Omega \vartheta_\epsilon^2 d_\epsilon\ dxds,$$

3.
$$\left|\int_0^t\int_\Omega [\partial_k^\epsilon\vartheta_\epsilon(\mathbf{g}^{k,\epsilon}\cdot\partial_t\mathbf{u}_\epsilon)d_\epsilon\ dxds\right| \overset{(5.2.7),(4.2.9),(4.2.10)}{\le}$$
$$\le \frac{C\eta_1}{C_1}\int_0^t\int_\Omega g_\epsilon^{kl}\partial_k^\epsilon K_{\frac{1}{2}}(\vartheta_\epsilon)\partial_l^\epsilon K_{\frac{1}{2}}(\vartheta_\epsilon)d_\epsilon\ dxds + C(\eta_1)\int_0^t\int_\Omega |\partial_t\mathbf{u}_\epsilon|^2 d_\epsilon\ dxds,$$
where η_1 is arbitrary small but positive, and

4.
$$\frac{1}{\epsilon}\left|\int_0^t\int_\Omega g_\epsilon^{kl}\vartheta_\epsilon^2\omega_{kl}^\epsilon(\partial_t\mathbf{u}_\epsilon)d_\epsilon\ dxds\right| \overset{(4.2.9),(5.1.20),(5.2.7)}{\le}$$
$$\le \frac{C\eta_2}{\epsilon^2}\int_0^t\int_\Omega B_\epsilon^{ijkl}\omega_{kl}^\epsilon(\partial_t\mathbf{u}_\epsilon)\omega_{ij}^\epsilon(\partial_t\mathbf{u}_\epsilon)d_\epsilon\ dxds +$$
$$+\eta_3 C(\eta_2)\int_0^t\int_\Omega g_\epsilon^{kl}\partial_k^\epsilon K_{\frac{1}{2}}(\vartheta_\epsilon)\partial_l^\epsilon K_{\frac{1}{2}}(\vartheta_\epsilon)d_\epsilon\ dxds +$$
$$+C(\eta_2,\eta_3)\int_0^t\left(\int_\Omega \vartheta_\epsilon^2 d_\epsilon\ dx\right)^2 ds,$$
where we have employed the inequality
$$\int_Q v^4\ dx \le \eta_3\int_Q |\nabla v^2|^2\ dx + C(\eta_3)\left(\int_Q v^2\ dx\right)^2$$
and where η_2, η_3 are sufficiently small.

We put now

$$y(t) := \frac{1}{2}\int_\Omega \left[|\partial_t \mathbf{u}_\epsilon(t)|^2 + \frac{1}{\epsilon^2}A_\epsilon^{ijkl}\omega_{kl}^\epsilon(\mathbf{u}_\epsilon(t))\omega_{ij}^\epsilon(\mathbf{u}_\epsilon(t)) + \vartheta_\epsilon^2(t)\right] d_\epsilon\, dx. \tag{5.2.28}$$

As a consequence of the above estimates we can rewrite (5.2.17) as

$$y(t) \le C\left(1 + \int_0^t y^2(s)\, ds\right).$$

Hence, employing Young's inequality, we get

$$y^2(t) \le C\left(1 + \left(\int_0^t y^2(s)\, ds\right)^2\right),$$

which can be rewritten as

$$z'(t) \le C(1 + z^2(t)),$$

where $z(t) = \int_0^t y^2(s)\, ds$. In view of nonnegativity $z(0)$, we get the estimate

$$z(t) \le \operatorname{tg}(\operatorname{arctg}(z(0)) + Ct) \tag{5.2.29}$$

for $t \in [0, T]$, where $T = T(C, z(0))$. As a result of (5.2.28) and (5.2.29), we have (passing to subsequences if necessary)

$$\mathbf{u}_\epsilon \overset{*}{\rightharpoonup} \mathbf{u} \text{ in } L^\infty(0, T; V(\Omega)),\ \vartheta_\epsilon \overset{*}{\rightharpoonup} \vartheta \text{ in } L^\infty(0, T; L^2(\Omega)), \tag{5.2.30}$$

$$\partial_t\mathbf{u}_\epsilon \overset{*}{\rightharpoonup} \partial_t\mathbf{u} \text{ in } L^\infty(0, T; L^2(\Omega)^3),\ \partial_t\mathbf{u}_\epsilon \rightharpoonup \partial_t\mathbf{u} \text{ in } L^2(0, T; V(\Omega)), \tag{5.2.31}$$

$$\frac{1}{\epsilon}\omega^\epsilon(\mathbf{u}_\epsilon) \overset{*}{\rightharpoonup} \zeta_{\mathbf{u}} \text{ in } L^\infty(0, T; L^2(\Omega)^9), \tag{5.2.32}$$

$$\frac{1}{\epsilon}\omega^\epsilon(\partial_t\mathbf{u}_\epsilon) \rightharpoonup \partial_t\zeta_{\mathbf{u}} \text{ in } L^2(0, T; L^2(\Omega)^9), \tag{5.2.33}$$

$$K_{\frac{1}{2}}(\vartheta_\epsilon) \rightharpoonup \overline{K_{\frac{1}{2}}(\vartheta)} \text{ in } L^2(0, T; W^{1,2}(\Omega)),\ \nabla\vartheta_\epsilon \rightharpoonup \nabla\vartheta \text{ in } L^2(0, T; L^2(\Omega)^3). \tag{5.2.34}$$

Since $\nabla_\epsilon\vartheta_\epsilon$ is bounded in $L^2(0, T; L^2(\Omega)^3)$ due to (5.2.29), we can deduce $\vartheta(x, t) = \vartheta(x_1, t)$. It follows from (5.1.95) that

$$\frac{1}{2\epsilon}(\partial_2\mathbf{u}_\epsilon \cdot \mathbf{b}_\epsilon - \partial_3\mathbf{u}_\epsilon \cdot \mathbf{n}_\epsilon) \overset{*}{\rightharpoonup} \phi_{\mathbf{u}} \text{ in } L^\infty(0, T; L^2(\Omega)), \tag{5.2.35}$$

$$\frac{1}{2\epsilon}(\partial_2\partial_t\mathbf{u}_\epsilon \cdot \mathbf{b}_\epsilon - \partial_3\partial_t\mathbf{u}_\epsilon \cdot \mathbf{n}_\epsilon) \rightharpoonup \partial_t\phi_{\mathbf{u}} \text{ in } L^2(0, T; L^2(\Omega)), \tag{5.2.36}$$

where $\langle \mathbf{u}, \phi_{\mathbf{u}}\rangle \in L^\infty(0, T; \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0, l))$, $\langle \partial_t\mathbf{u}, \partial_t\phi_{\mathbf{u}}\rangle \in L^2(0, T; \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0, l))$ and

$$\partial_1\mathbf{u}\cdot\mathbf{t} = 0 \text{ a.e. in } (0, l) \times (0, T), \tag{5.2.37}$$

$$\partial_1\partial_t\mathbf{u}\cdot\mathbf{t} = 0 \text{ a.e. in } (0, l) \times (0, T), \tag{5.2.38}$$

$$\partial_1\mathbf{u}_*\cdot\mathbf{t} = \partial_3\zeta_{\mathbf{u},12} - \partial_2\zeta_{\mathbf{u},13} \text{ in } L^\infty(0, T; L^2(0, l; [W_0^{1,2}(S)]')), \tag{5.2.39}$$

$$\partial_1\partial_t\mathbf{u}_*\cdot\mathbf{t} = \partial_3\partial_t\zeta_{\mathbf{u},12} - \partial_2\partial_t\zeta_{\mathbf{u},13} \text{ in } L^2(0, T; L^2(0, l; [W_0^{1,2}(S)]')), \tag{5.2.40}$$

$$\partial_1\mathbf{u}_*\cdot\mathbf{n} = -\partial_3\zeta_{\mathbf{u},11} \text{ a.e. in } (0, l) \times (0, T), \tag{5.2.41}$$

$$\partial_1\partial_t\mathbf{u}_*\cdot\mathbf{n} = -\partial_3\partial_t\zeta_{\mathbf{u},11} \text{ a.e. in } (0, l) \times (0, T), \tag{5.2.42}$$

$$\partial_1\mathbf{u}_*\cdot\mathbf{b} = \partial_2\zeta_{\mathbf{u},11} \text{ a.e. in } (0, l) \times (0, T), \tag{5.2.43}$$

$$\partial_1\partial_t\mathbf{u}_*\cdot\mathbf{b} = \partial_2\partial_t\zeta_{\mathbf{u},11} \text{ a.e. in } (0, l) \times (0, T). \tag{5.2.44}$$

We can now multiply (5.2.15) with ϵ^2 and pass to the limit using the convergences (5.2.30)–(5.2.34). Then we get

$$\int_\Omega (A^{ijkl}\zeta_{\mathbf{u},kl}(t) + B^{ijkl}\partial_t\zeta_{\mathbf{u},kl}(t))\theta^0_{ij}(\mathbf{v})\ dx = 0 \tag{5.2.45}$$

for all $\mathbf{v} \in L^2(0,l;W^{1,2}(S)^3)$ and for a.a. $t \in (0,T)$, where the tensor $\theta^0(\mathbf{v})$ is defined by

$$\theta^0(\mathbf{v}) := \begin{pmatrix} 0 & \frac{\partial_2\mathbf{v}\cdot\mathbf{t}}{2} & \frac{\partial_3\mathbf{v}\cdot\mathbf{t}}{2} \\ \frac{\partial_2\mathbf{v}\cdot\mathbf{t}}{2} & \partial_2\mathbf{v}\cdot\mathbf{n} & \frac{\partial_2\mathbf{v}\cdot\mathbf{b}+\partial_3\mathbf{v}\cdot\mathbf{n}}{2} \\ \frac{\partial_3\mathbf{v}\cdot\mathbf{t}}{2} & \frac{\partial_2\mathbf{v}\cdot\mathbf{b}+\partial_3\mathbf{v}\cdot\mathbf{n}}{2} & \partial_3\mathbf{v}\cdot\mathbf{b} \end{pmatrix}. \tag{5.2.46}$$

We put now $\mathbf{v} = v_1(x_1,t)v(x_2,x_3)\mathbf{t}$ in (5.2.45). Then

$$\mu\int_S[\zeta_{\mathbf{u},12}\partial_2 v + \zeta_{\mathbf{u},13}\partial_3 v]\ dx_2dx_3 + \mu_v\int_S[\partial_t\zeta_{\mathbf{u},12}\partial_2 v + \partial_t\zeta_{\mathbf{u},13}\partial_3 v]\ dx_2dx_3 = 0.$$

If we put $v = x_2$, $v = x_3$, $v = x_2^2$, $v = x_3^2$, and $v = x_2x_3$, we get

$$\int_S[\mu\zeta_{\mathbf{u},12} + \mu_v\partial_t\zeta_{\mathbf{u},12}]\ dx_2dx_3 = \int_S[\mu\zeta_{\mathbf{u},12} + \mu_v\partial_t\zeta_{\mathbf{u},12}]x_2\ dx_2dx_3 =$$

$$= \int_S[\mu\zeta_{\mathbf{u},13} + \mu_v\partial_t\zeta_{\mathbf{u},13}]\ dx_2dx_3 = \int_S[\mu\zeta_{\mathbf{u},13} + \mu_v\partial_t\zeta_{\mathbf{u},13}]x_3\ dx_2dx_3 =$$

$$= \int_S[(\mu\zeta_{\mathbf{u},12} + \mu_v\partial_t\zeta_{\mathbf{u},12})x_3 + (\mu\zeta_{\mathbf{u},13} + \mu_v\partial_t\zeta_{\mathbf{u},13})x_2]\ dx_2dx_3 = 0. \tag{5.2.47}$$

Similarly we can put $\mathbf{v} = v_1(x_1,t)v(x_2,x_3)\mathbf{n}$ and $\mathbf{v} = v_1(x_1,t)v(x_2,x_3)\mathbf{b}$ in (5.2.45), which leads to

$$2\mu\int_S[\zeta_{\mathbf{u},22}\partial_2 v + \zeta_{\mathbf{u},23}\partial_3 v]\ dx_2dx_3 + \lambda\int_S[\zeta_{\mathbf{u},11} + \zeta_{\mathbf{u},22} + \zeta_{\mathbf{u},33}]\partial_2 v\ dx_2dx_3 +$$

$$+2\mu_v\int_S[\partial_t\zeta_{\mathbf{u},22}\partial_2 v + \partial_t\zeta_{\mathbf{u},23}\partial_3 v]\ dx_2dx_3 +$$

$$+\lambda_v\int_S[\partial_t\zeta_{\mathbf{u},11} + \partial_t\zeta_{\mathbf{u},22} + \partial_t\zeta_{\mathbf{u},33}]\partial_2 v\ dx_2dx_3 = 0$$

and

$$2\mu\int_S[\zeta_{\mathbf{u},23}\partial_2 v + \zeta_{\mathbf{u},33}\partial_3 v]\ dx_2dx_3 + \lambda\int_S[\zeta_{\mathbf{u},11} + \zeta_{\mathbf{u},22} + \zeta_{\mathbf{u},33}]\partial_3 v\ dx_2dx_3 +$$

$$+2\mu_v\int_S[\partial_t\zeta_{\mathbf{u},23}\partial_2 v + \partial_t\zeta_{\mathbf{u},33}\partial_3 v]\ dx_2dx_3 +$$

$$+\lambda_v\int_S[\partial_t\zeta_{\mathbf{u},11} + \partial_t\zeta_{\mathbf{u},22} + \partial_t\zeta_{\mathbf{u},33}]\partial_3 v\ dx_2dx_3 = 0.$$

We put again $v = x_2$, $v = x_3$, $v = x_2^2$, $v = x_3^2$, and $v = x_2x_3$. Then

$$\int_S[\mu\zeta_{\mathbf{u},23} + \mu_v\partial_t\zeta_{\mathbf{u},23}]\ dx_2dx_3 = \int_S[\mu\zeta_{\mathbf{u},23} + \mu_v\partial_t\zeta_{\mathbf{u},23}]x_2\ dx_2dx_3 =$$

$$= \int_S[\mu\zeta_{\mathbf{u},23} + \mu_v\partial_t\zeta_{\mathbf{u},23}]x_3\ dx_2dx_3 = 0, \tag{5.2.48}$$

$$2\int_S[\mu\zeta_{\mathbf{u},22} + \mu_v\partial_t\zeta_{\mathbf{u},22}]\ dx_2dx_3 +$$

$$+\int_S [\lambda(\zeta_{\mathbf{u},11}+\zeta_{\mathbf{u},22}+\zeta_{\mathbf{u},33})+\lambda_v(\partial_t\zeta_{\mathbf{u},11}+\partial_t\zeta_{\mathbf{u},22}+\partial_t\zeta_{\mathbf{u},33})]\ dx_2dx_3 =$$

$$= 2\int_S [\mu\zeta_{\mathbf{u},22}+\mu_v\partial_t\zeta_{\mathbf{u},22}]x_2\ dx_2dx_3+$$

$$+\int_S [\lambda(\zeta_{\mathbf{u},11}+\zeta_{\mathbf{u},22}+\zeta_{\mathbf{u},33})+\lambda_v(\partial_t\zeta_{\mathbf{u},11}+\partial_t\zeta_{\mathbf{u},22}+\partial_t\zeta_{\mathbf{u},33})]x_2\ dx_2dx_3 =$$

$$= 2\int_S [\mu\zeta_{\mathbf{u},22}+\mu_v\partial_t\zeta_{\mathbf{u},22}]x_3\ dx_2dx_3+$$

$$+\int_S [\lambda(\zeta_{\mathbf{u},11}+\zeta_{\mathbf{u},22}+\zeta_{\mathbf{u},33})+\lambda_v(\partial_t\zeta_{\mathbf{u},11}+\partial_t\zeta_{\mathbf{u},22}+\partial_t\zeta_{\mathbf{u},33})]x_3\ dx_2dx_3 = 0, \quad (5.2.49)$$

and

$$2\int_S [\mu\zeta_{\mathbf{u},33}+\mu_v\partial_t\zeta_{\mathbf{u},33}]\ dx_2dx_3+$$

$$+\int_S [\lambda(\zeta_{\mathbf{u},11}+\zeta_{\mathbf{u},22}+\zeta_{\mathbf{u},33})+\lambda_v(\partial_t\zeta_{\mathbf{u},11}+\partial_t\zeta_{\mathbf{u},22}+\partial_t\zeta_{\mathbf{u},33})]\ dx_2dx_3 =$$

$$= 2\int_S [\mu\zeta_{\mathbf{u},33}+\mu_v\partial_t\zeta_{\mathbf{u},33}]x_2\ dx_2dx_3+$$

$$+\int_S [\lambda(\zeta_{\mathbf{u},11}+\zeta_{\mathbf{u},22}+\zeta_{\mathbf{u},33})+\lambda_v(\partial_t\zeta_{\mathbf{u},11}+\partial_t\zeta_{\mathbf{u},22}+\partial_t\zeta_{\mathbf{u},33})]x_2\ dx_2dx_3 =$$

$$= 2\int_S [\mu\zeta_{\mathbf{u},33}+\mu_v\partial_t\zeta_{\mathbf{u},33}]x_3\ dx_2dx_3+$$

$$+\int_S [\lambda(\zeta_{\mathbf{u},11}+\zeta_{\mathbf{u},22}+\zeta_{\mathbf{u},33})+\lambda_v(\partial_t\zeta_{\mathbf{u},11}+\partial_t\zeta_{\mathbf{u},22}+\partial_t\zeta_{\mathbf{u},33})]x_3\ dx_2dx_3 = 0. \quad (5.2.50)$$

Let us define vector $\boldsymbol{\eta} := (\varsigma_{\mathbf{u},12}, \varsigma_{\mathbf{u},13})$, where $\varsigma_{\mathbf{u},1i} = \mu\zeta_{\mathbf{u},1i} + \mu_v\partial_t\zeta_{\mathbf{u},1i}$, $i = 2, 3$. We put further $\mathbf{v} = \varphi\mathbf{t}$, $\varphi \in L^2(0,l;W^{1,2}(S))$ in (5.2.45). Then (5.2.39)–(5.2.40) and (5.2.45) can be rewritten as

$$\int_\Omega \boldsymbol{\eta}(t)\cdot\nabla_{23}\varphi\ dx = 0,\ \forall\varphi \in L^2(0,l;W^{1,2}(S)), \quad (5.2.51)$$

$$\int_\Omega \boldsymbol{\eta}(t)\cdot \mathrm{rot}_{23}\psi\ dx = \int_\Omega (\mu_v\partial_t\partial_1\mathbf{u}_*\cdot\mathbf{t} + \mu\partial_1\mathbf{u}_*\cdot\mathbf{t})\psi\ dx,\ \forall\psi\in W_0^{1,2}(\Omega), \quad (5.2.52)$$

for a.a. $t \in (0,T)$, where we have denoted $\nabla_{23}\varphi = (\partial_2\varphi, \partial_3\varphi)$ and $\mathrm{rot}_{23}\psi = (-\partial_3\psi, \partial_2\psi)$. According to Lemma 5.1.10 there exists a unique solution $\boldsymbol{\eta} \in L^\infty(0,T;L^2(\Omega)^2)$ to (5.2.51)–(5.2.52) having the form

$$\boldsymbol{\eta} = (\varsigma_{\mathbf{u},12}, \varsigma_{\mathbf{u},13}) = -\frac{1}{2}(\mu_v\partial_t\partial_1\mathbf{u}_*\cdot\mathbf{t} + \mu\partial_1\mathbf{u}_*\cdot\mathbf{t})(\partial_2 p - x_3, \partial_3 p + x_2), \quad (5.2.53)$$

where the function $p \in W^{1,2}(S)$ is the unique solution to the Neumann problem (5.1.85).

At the end of this section, we derive equations (5.2.22) and (5.2.23). Let $\langle\mathbf{v}, \phi_{\mathbf{v}}\rangle$ be an arbitrary couple of functions from the space $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ and the couples $\langle\mathbf{v}_\epsilon, \phi_{\mathbf{v}_\epsilon}\rangle \in \mathcal{V}_0^{\mathbf{t}_\epsilon,\mathbf{n}_\epsilon,\mathbf{b}_\epsilon}(0,l)$ its smooth approximations (see (4.4.5)–(4.4.7)). As in the previous section, we define the functions

$$\begin{aligned}\mathbf{w}_\epsilon(x_1,x_2,x_3) \quad &:= \quad -\Big((\mathbf{v}'_\epsilon(x_1)\cdot\mathbf{n}_\epsilon(x_1))x_2 + (\mathbf{v}'_\epsilon(x_1)\cdot\mathbf{b}_\epsilon(x_1))x_3\Big)\mathbf{t}_\epsilon(x_1) - \\ &\quad -x_3\phi_{\mathbf{v}_\epsilon}(x_1)\mathbf{n}_\epsilon(x_1) + x_2\phi_{\mathbf{v}_\epsilon}(x_1)\mathbf{b}_\epsilon(x_1), \quad (5.2.54)\\ \widehat{\mathbf{v}}_\epsilon(x_1,x_2,x_3) \quad &:= \quad \mathbf{v}_\epsilon(x_1) + \epsilon\mathbf{w}_\epsilon(x_1,x_2,x_3)\ \text{for}\ (x_1,x_2,x_3)\in\Omega, \quad (5.2.55)\end{aligned}$$

and derive relations (5.1.108)–(5.1.113) that provide us with

$$\Upsilon_{ij}(\mathbf{v}_{*,\epsilon}) \to \Upsilon_{ij}(\mathbf{v}_*) \text{ in } L^2(\Omega),\ i,j=1,2,3, \tag{5.2.56}$$

$$\|B_\epsilon\|_2 = \|B_\epsilon^{11}\|_2 \le C\epsilon^{2(1-r)},\ r \in \left(0, \frac{1}{3}\right), \tag{5.2.57}$$

$$\widehat{\mathbf{v}}_\epsilon \to \mathbf{v} \text{ in } W^{1,2}(\Omega)^3 \tag{5.2.58}$$

for $\epsilon \to 0$.

We put $\mathbf{v} = \varphi\widehat{\mathbf{v}}_\epsilon$ in (5.2.15), where $\widehat{\mathbf{v}}_\epsilon$ is defined by (5.2.54)–(5.2.55) and $\varphi \in C^\infty([0,T])$, $\varphi(T)=0$. We further denote $\overline{v}^\varphi := \int_0^T \varphi v\ dt$. Then it follows from (5.2.20)–(5.2.21), (5.2.30)–(5.2.36), (5.1.108)–(5.1.113), and (5.2.56)–(5.2.58) that we can pass from the equation

$$\begin{aligned} &-\int_\Omega \overline{\partial_t \mathbf{u}_\epsilon}^{\partial_t\varphi} \cdot \widehat{\mathbf{v}}_\epsilon d_\epsilon\ dx + \int_\Omega \varphi(0)\mathbf{v}_{0,\epsilon}\cdot\widehat{\mathbf{v}}_\epsilon d_\epsilon\ dx + \\ &+\int_\Omega B_\epsilon^{ijkl}\frac{1}{\epsilon}\omega_{kl}^\epsilon(\overline{\partial_t\mathbf{u}_\epsilon}^\varphi)\frac{1}{\epsilon}\omega_{ij}^\epsilon(\widehat{\mathbf{v}}_\epsilon)d_\epsilon\ dx + \int_\Omega A_\epsilon^{ijkl}\frac{1}{\epsilon}\omega_{kl}^\epsilon(\overline{\mathbf{u}_\epsilon}^\varphi)\frac{1}{\epsilon}\omega_{ij}^\epsilon(\widehat{\mathbf{v}}_\epsilon)d_\epsilon\ dx = \\ &= \int_\Omega \overline{\mathbf{f}_\epsilon}^\varphi \cdot \widehat{\mathbf{v}}_\epsilon d_\epsilon\ dx - \upsilon(3\lambda+2\mu)\int_\Omega \overline{\partial_k^\epsilon\vartheta_\epsilon}^\varphi(\mathbf{g}^{k,\epsilon}\cdot\widehat{\mathbf{v}}_\epsilon)d_\epsilon\ dx \end{aligned} \tag{5.2.59}$$

to

$$\begin{aligned} &-|S|\int_0^l \overline{\partial_t\mathbf{u}}^{\partial_t\varphi}\cdot\mathbf{v}\ dx_1 + |S|\int_0^l \varphi(0)\mathbf{v}_0\cdot\mathbf{v}\ dx_1 + \int_\Omega B^{ijkl}\overline{\partial_t\zeta_{\mathbf{u},kl}}^\varphi\Upsilon_{ij}(\mathbf{v}_*)\ dx + \\ &+\int_\Omega A^{ijkl}\overline{\zeta_{\mathbf{u},kl}}^\varphi\Upsilon_{ij}(\mathbf{v}_*)\ dx = \int_\Omega \overline{\mathbf{f}}^\varphi\cdot\mathbf{v}\ dx - \int_\Omega \overline{d_k}^\varphi(\mathbf{g}^k\cdot\mathbf{v})\ dx \end{aligned} \tag{5.2.60}$$

for any $\langle \mathbf{v}, \phi_\mathbf{v}\rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$, where

$$\mathbf{g}^1 = \mathbf{t},\ \mathbf{g}^2 = \mathbf{n},\ \mathbf{g}^3 = \mathbf{b},$$

and

$$\nabla_\epsilon\vartheta_\epsilon \rightharpoonup \mathbf{d} \text{ in } L^2(0,T;L^2(\Omega)^3) \tag{5.2.61}$$

as a consequence of (5.2.7), (5.2.17), and (5.2.27)–(5.2.29).

We now pay attention to the heat equation. Since the estimate

$$\left\| v - \frac{1}{|S|}\int_S v\ dx_2dx_3 \right\|_{p,S} \le C(p)\|\nabla_{23}v\|_{2,S} \tag{5.2.62}$$

holds for any $p \in [1,\infty)$ and $v \in W^{1,2}(S)$, it follows from the definition of $\nabla_\epsilon\vartheta_\epsilon$ that

$$\left\| \vartheta_\epsilon - \frac{1}{|S|}\int_S \vartheta_\epsilon\ dx_2dx_3 \right\|_{L^2((0,l)\times(0,T);L^p(S))} \to 0 \tag{5.2.63}$$

for any $p \in [1,\infty)$.

We put now $\psi = \psi(x_1,t)$ in (5.2.16). Then we can deduce that

$$\partial_t\int_S \vartheta_\epsilon\ dx_2dx_3 \text{ is bounded in } L^{\frac{\alpha+1}{\alpha}}(0,T;[W^{1,\frac{3(\alpha+1)}{\alpha+2}}(0,l)]'). \tag{5.2.64}$$

Since (5.2.34) ensures that

$$\int_S \vartheta_\epsilon\ dx_2dx_3 \text{ is bounded in } L^2(0,T;W^{1,2}(0,l)),$$

then (passing to a subsequence if necessary)

$$\int_S \vartheta_\epsilon \, dx_2 dx_3 \to \int_S \vartheta \, dx_2 dx_3 \text{ in } L^2(0,T;C([0,l])). \tag{5.2.65}$$

By virtue of (5.2.63) we can conclude

$$\|\vartheta_\epsilon - \vartheta\|_{L^2((0,l)\times(0,T);L^p(S))} \to 0 \tag{5.2.66}$$

for any $p \in [1,\infty)$, because $\vartheta = \vartheta(x_1,t)$. This implies $\overline{\kappa^{\frac{1}{2}}(\vartheta)} = \kappa^{\frac{1}{2}}(\vartheta)$ and $\overline{K_{\frac{1}{2}}(\vartheta)} = K_{\frac{1}{2}}(\vartheta)$. In addition, we can prove that

$$\kappa^{\frac{1}{2}}(\vartheta_\epsilon) \to \kappa^{\frac{1}{2}}(\vartheta) \text{ and } K_{\frac{1}{2}}(\vartheta_\epsilon) \to K_{\frac{1}{2}}(\vartheta) \text{ in } L^1(\Omega \times (0,T)). \tag{5.2.67}$$

We demonstrate it only for $K_{\frac{1}{2}}(\vartheta_\epsilon)$. Let us recall that (5.2.17) and (5.2.34) imply that

$$\left\{ \int_0^T \left(\int_\Omega \vartheta_\epsilon^{3(\alpha+1)} \, dx \right)^{\frac{1}{3}} \, dt \right\}_{\epsilon \in (0,1)} \text{ is bounded }. \tag{5.2.68}$$

Applying the Mean Value Theorem in the following estimates

$$\int_0^T \int_0^l \int_S |K_{\frac{1}{2}}(\vartheta_\epsilon) - K_{\frac{1}{2}}(\vartheta)| \, dxdt \le \int_0^T \int_0^l \int_S \kappa^{\frac{1}{2}}(\vartheta_\epsilon)|\vartheta_\epsilon - \vartheta| \, dxdt +$$

$$+ \int_0^T \int_0^l \int_S \kappa^{\frac{1}{2}}(\vartheta)|\vartheta_\epsilon - \vartheta| \, dxdt = I_1 + I_2,$$

we can see that it is enough to treat only I_1 because I_2 can be estimated similarly. Thus

$$I_1 \le C \int_0^T \int_0^l \int_S \vartheta_\epsilon^{\frac{\alpha-1}{2}} |\vartheta_\epsilon - \vartheta| \, dxdt + C \int_0^T \int_0^l \int_S |\vartheta_\epsilon - \vartheta| \, dxdt \le$$

$$\le C \sqrt{\int_0^T \int_0^l \left(\int_S |\vartheta_\epsilon - \vartheta|^{\frac{6}{5}} \, dx_2 dx_3 \right)^{\frac{5}{3}} \, dx_1 dt} \sqrt{\int_0^T \left(\int_\Omega \vartheta_\epsilon^{3(\alpha-1)} \, dx \right)^{\frac{1}{3}} \, dt}$$

$$+ C\|\vartheta_\epsilon - \vartheta\|_{1,\Omega\times(0,T)}.$$

The rest follows from (5.2.66) and (5.2.68). To pass to the limit in the third term in (5.2.16) we must check

$$\kappa^{\frac{1}{2}}(\vartheta_\epsilon) \to \kappa^{\frac{1}{2}}(\vartheta) \text{ in } L^2(\Omega \times (0,T)). \tag{5.2.69}$$

Because of (5.2.67), we can restrict ourselves to the proof of the boundedness of $\{\kappa^{\frac{1}{2}}(\vartheta_\epsilon)\}_{\epsilon\in(0,1)}$ in $L^q(\Omega \times (0,T))$ for some $q > 2$. Due to (5.2.7), it is enough to check

$$\int_0^T \int_\Omega \vartheta_\epsilon^{\frac{\alpha-1}{2}q} \, dxdt = \int_0^T \int_\Omega \vartheta_\epsilon^{\frac{\alpha-1}{2}q-\gamma} \vartheta_\epsilon^\gamma \, dxdt \le$$

$$\le \int_0^T \left(\int_\Omega \vartheta_\epsilon^2 \, dx \right)^{\frac{\gamma}{2}} \left(\int_\Omega \vartheta_\epsilon^{(\frac{\alpha-1}{2}q-\gamma)\frac{2}{2-\gamma}} \, dx \right)^{\frac{2-\gamma}{2}} \, dt \overset{(5.2.30),\,(5.2.68)}{\le} C, \tag{5.2.70}$$

where we have put $\gamma = \frac{4}{3}$ and $q = \frac{2\alpha+\frac{14}{3}}{\alpha-1} > 2$. It is an easy matter to check that $\frac{\alpha-1}{2}q - \frac{4}{3} > 0$ for $\alpha > \frac{7}{3}$.

We put now $\psi = \psi(x_1,t)$ in (5.2.16). Then (5.2.21), (5.2.30), (5.2.33), (5.2.34), (5.2.61), (5.2.66), and (5.2.69) enable us to pass to the limit in (5.2.16) and to derive

$$-|S| \int_0^T \int_0^l \vartheta \partial_t \psi \, dx_1 dt + |S| \int_0^l \vartheta_0 \psi(\cdot,0) \, dx_1 + |S| \int_0^T \int_0^l \kappa(\vartheta) \partial_1 \vartheta \partial_1 \psi \, dx_1 dt +$$

$$+\upsilon(3\lambda+2\mu)\int_0^T\int_0^l \vartheta\psi\int_S \partial_t\zeta_{\mathbf{u},kk}\ dx_2dx_3dx_1dt=\int_0^T\int_0^l \check{h}\psi\ dx_1dt \qquad (5.2.71)$$

for any $\psi\in C^\infty([0,T];C^\infty([0,l]))$, where $\psi(\cdot,T)=0$ and $\check{h}:=\frac{1}{|S|}\int_S h\ dx_2dx_3$.

At the end, we will pay attention to the unknown terms in (5.2.60) and (5.2.71). Let us start with $\mathbf{d}$ (see (5.2.60 and (5.2.61)). (5.2.34) implies that $d_1=\partial_1\vartheta(x_1,t)$. Let us take $\frac{1}{\epsilon}K_{\frac{1}{2}}(\vartheta_\epsilon)$. Then (5.2.17) implies

$$\frac{1}{\epsilon}\nabla_{23}K_{\frac{1}{2}}(\vartheta_\epsilon)\rightharpoonup\nabla_{23}\hat{d}\ \text{in}\ L^2(0,T;L^2(\Omega)^2). \qquad (5.2.72)$$

We multiply now (5.2.16) with ϵ and use (5.2.69). Then after the limit passage we get

$$\int_0^T\int_\Omega \kappa^{\frac{1}{2}}(\vartheta)(\partial_2\hat{d}\partial_2\psi+\partial_3\hat{d}\partial_3\psi)\ dxdt=0,$$

which implies due to (5.2.7)

$$\int_S \partial_2\hat{d}\partial_2\psi+\partial_3\hat{d}\partial_3\psi\ dx_2dx_3dt=0\ \text{a.e. in}\ (0,l)\times(0,T)$$

for any $\psi\in C^1(\bar{S})$. By virtue of (5.2.72) and density of $C^1(\bar{S})$ in $W^{1,2}(S)$, we thus have $\nabla_{23}\hat{d}=0$. (5.2.61), (5.2.69), and (5.2.72) imply

$$\nabla_{23}\hat{d}=\kappa^{\frac{1}{2}}(\vartheta)\mathbf{d}_{23},\ \text{where}\ \mathbf{d}_{23}:=(d_2,d_3).$$

Thus $d_2=d_3=0$.

Relations (5.2.41)–(5.2.44) enable us to express $\zeta_{\mathbf{u},11}$ and $\partial_t\zeta_{\mathbf{u},11}$ as

$$\zeta_{\mathbf{u},11}=Q_0+(\partial_1\mathbf{u}_*\cdot\mathbf{b})x_2-(\partial_1\mathbf{u}_*\cdot\mathbf{n})x_3, \qquad (5.2.73)$$

$$\partial_t\zeta_{\mathbf{u},11}=\partial_tQ_0+(\partial_t\partial_1\mathbf{u}_*\cdot\mathbf{b})x_2-(\partial_t\partial_1\mathbf{u}_*\cdot\mathbf{n})x_3, \qquad (5.2.74)$$

where $Q_0=Q_0(x_1,t)$. The next unknown terms from (5.2.60) can be expressed by using (5.1.108)–(5.1.113) and (4.2.56)–(5.2.58) as

$$\int_\Omega B^{ijkl}\overline{\partial_t\zeta_{\mathbf{u},kl}}^\varphi\Upsilon_{ij}(\mathbf{v}_*)\ dx+\int_\Omega A^{ijkl}\overline{\zeta_{\mathbf{u},kl}}^\varphi\Upsilon_{ij}(\mathbf{v}_*)\ dx=$$

$$=\int_0^T\varphi\int_\Omega[(2\mu_v\partial_t\zeta_{\mathbf{u},11}+2\mu\zeta_{\mathbf{u},11})+\lambda_v(\partial_t\zeta_{\mathbf{u},11}+\partial_t\zeta_{\mathbf{u},22}+\partial_t\zeta_{\mathbf{u},33})+$$

$$+\lambda(\zeta_{\mathbf{u},11}+\zeta_{\mathbf{u},22}+\zeta_{\mathbf{u},33})]\Upsilon_{11}(\mathbf{v}_*)\ dxdt+$$

$$+4\int_0^T\varphi\int_\Omega[(\mu_v\partial_t\zeta_{\mathbf{u},12}+\mu\zeta_{\mathbf{u},12})\Upsilon_{12}(\mathbf{v}_*)+(\mu_2\partial_t\zeta_{\mathbf{u},13}+\mu\zeta_{\mathbf{u},13})\Upsilon_{13}(\mathbf{v}_*)]\ dxdt=$$

$$=I_1+I_2. \qquad (5.2.75)$$

To treat I_2 we use (5.2.53). We now pay attention to I_1. Let us denote

$$y_1(t):=\int_S\zeta_{\mathbf{u},kk}\ dx_2dx_3,\ \ f_1(t):=\mu_v\frac{d}{dt}\int_S\zeta_{\mathbf{u},11}\ dx_2dx_3+\mu\int_S\zeta_{\mathbf{u},11}\ dx_2dx_3, \qquad (5.2.76)$$

and similarly

$$y_j(t):=\int_S(\zeta_{\mathbf{u},11}+\zeta_{\mathbf{u},22}+\zeta_{\mathbf{u},33})x_j\ dx_2dx_3,\ \ j=2,3, \qquad (5.2.77)$$

$$f_j(t):=\mu_v\frac{d}{dt}\int_S\zeta_{\mathbf{u},11}x_j\ dx_2dx_3+\mu\int_S\zeta_{\mathbf{u},11}x_j\ dx_2dx_3,\ \ j=2,3. \qquad (5.2.78)$$

As a consequence of (5.2.49)–(5.2.50) we get the equations

$$(\lambda_v + \mu_v) y_j'(t) + (\lambda + \mu) y_j(t) = f_j(t),\ j = 1, 2, 3. \tag{5.2.79}$$

The only solutions to (4.2.79) are (see (5.2.18), (5.2.19), (5.2.32), (5.2.33))

$$y_1(t) = e^{-Ft} \int_S \zeta_{\mathbf{u}_0,kk}\ dx_2dx_3 + \frac{1}{\lambda_v + \mu_v} e^{-Ft} \int_0^t f_1(s) e^{Fs}\ ds, \tag{5.2.80}$$

$$y_2(t) = e^{-Ft} \int_S (\zeta_{\mathbf{u}_0,11} + \zeta_{\mathbf{u}_0,22} + \zeta_{\mathbf{u}_0,33}) x_2\ dx_2dx_3 + \frac{1}{\lambda_v + \mu_v} e^{-Ft} \int_0^t f_2(s) e^{Fs}\ ds, \tag{5.2.81}$$

and

$$y_3(t) = e^{-Ft} \int_S (\zeta_{\mathbf{u}_0,11} + \zeta_{\mathbf{u}_0,22} + \zeta_{\mathbf{u}_0,33}) x_3\ dx_2dx_3 + \frac{1}{\lambda_v + \mu_v} e^{-Ft} \int_0^t f_3(s) e^{Fs}\ ds. \tag{5.2.82}$$

Using (4.2.1) together with the functions y_j, $j = 1, 2, 3$, enable us to express I_1 from (5.2.75) and to get (5.2.22).

It remains to express the last unknown term in (5.2.71). We multiply (5.2.15) with ϵ and put $\mathbf{v} = \mathbf{v}(x_1, t)$, where $\mathbf{v} \in C_0^\infty(0, T; C_0^\infty([0, l])^3)$. Given (5.1.5)–(5.1.8) and (5.2.47), we get

$$\int_0^T \int_0^l (\partial_1 \mathbf{v} \cdot \mathbf{t}) \Big[2\mu_v \frac{d}{dt} \int_S \zeta_{\mathbf{u},11}\ dx_2dx_3 + 2\mu \int_S \zeta_{\mathbf{u},11}\ dx_2dx_3\ dx_1dt + \\ + \lambda_v \int_S \partial_t \zeta_{\mathbf{u},kk}\ dx_2dx_3 + \lambda \int_S \zeta_{\mathbf{u},kk}\ dx_2dx_3 \Big]\ dx_1dt = 0. \tag{5.2.83}$$

By virtue of properties of $\mathbf{t}$ (see assumptions of Theorem 5.2.2), we arrive at

$$2\mu_v \frac{d}{dt} \int_S \zeta_{\mathbf{u},11}\ dx_2dx_3 + 2\mu \int_S \zeta_{\mathbf{u},11}\ dx_2dx_3\ dx_1dt + \\ + \lambda_v \int_S \partial_t \zeta_{\mathbf{u},kk}\ dx_2dx_3 + \lambda \int_S \zeta_{\mathbf{u},kk}\ dx_2dx_3 = 0 \text{ a.e. in } (0, l) \times (0, T). \tag{5.2.84}$$

Because of (5.2.76), we can rewrite (5.2.84) as

$$\lambda_v y_1'(t) + \lambda y_1(t) = -2 f_1(t), \tag{5.2.85}$$

which, together with (5.2.79) for $j = 1$, leads to the equation

$$(2\mu_v + 3\lambda_v) y_1'(t) + (2\mu + 3\lambda) y_1(t) = 0. \tag{5.2.86}$$

The rest of the proof is obvious. □

6 Fluids

6.1 Introduction

In the previous sections, we could see how the dimension reduction works when we pass from a three-dimensional space to a one-dimensional in the case we have a curved domain. In this section, we pay attention to fluids and we show how a similar approach works in the case pass from a three-dimensional space to a two-dimensional. In the case of elastic materials, we speak about shells and the problems related to the dimension reduction were solved in [44] and under lower regularity assumptions in [28]. Application of the approach to compressible fluids was untried and was published for the first time in [13].

In the section, we pay attention to compressible fluids and thus let us briefly remind some of the results from the introduction. Compressible fluids are usually modeled by the Navier-Stokes equations. The respective system of equations is highly nonlinear and the problem of the existence of a solution was very hard to tackle. The first existence results can be found in [118]. Global existence theory for the full Navier-Stokes-Fourier system was extensively studied in [77] and for steady and unsteady isentropic flows we refer the reader to [161]. In the thesis, we pay attention to compressible, nonlinearly viscous isothermal fluids. The proof of the existence of a solution was given by Mamontov in [126] and [127]. As in the case of elastic materials, there is no unique approach to dimension reduction. The first attempt related to fluids was done in [154], where three-dimensional steady Navier- Stokes equations were asymptotically analyzed and the proof is based on the asymptotic expansion. But in the case of steady Navier-Stokes equations, we can also use a more direct approach (i.e. without any asymptotic expansion) as it was demonstrated in [213].

The next step is to study nonsteady Navier-Stokes equations for incompressible fluids. They were simplified into a lower-dimensional model in [90]. As in the case of elastic materials, we can derive two- or one-dimensional models, which was demonstrated in [212] and [12], respectively. Considering long thin pipes, it was shown in [22] that weak solutions of three-dimensional Navier-Stokes equations for barotropic flows converge to strong solutions of the respective one-dimensional system as the three-dimensional models converge to the one-dimensional model. The same result was achieved also for the full Navier-Stokes-Fourier system [29]. Similarly, the dimension reduction of barotropic Navier-Stokes equations from the three-dimensional system to the two-dimensional system was conducted in [124].

To ensure the consistency of the thesis we introduce the results from [13]. The deformation of the domain brings again new difficulties into the asymptotic analysis because the deformation affects the limit equations in a non-trivial way.

We study the asymptotic behavior of compressible fluids in thin domains $\widetilde{\Omega}_\epsilon \subset \mathbb{R}^3$. The motion of the compressible fluids is determined by the velocity $\widetilde{\mathbf{u}}_\epsilon$ and the density $\widetilde{\rho}_\epsilon$. The time evolution of $\widetilde{\mathbf{u}}_\epsilon$ and $\widetilde{\rho}_\epsilon$ is governed by the continuity and momentum equations

$$\begin{aligned} \partial_t \widetilde{\rho}_\epsilon + \widetilde{\operatorname{div}}\ (\widetilde{\rho}_\epsilon \widetilde{\mathbf{u}}_\epsilon) &= 0, && (6.1.1)\\ \partial_t \left(\widetilde{\rho}_\epsilon \widetilde{\mathbf{u}}_\epsilon\right) + \widetilde{\operatorname{div}}\ (\widetilde{\rho}_\epsilon \widetilde{\mathbf{u}}_\epsilon \otimes \widetilde{\mathbf{u}}_\epsilon) + \widetilde{\nabla} \widetilde{p}_\epsilon &= \widetilde{\operatorname{div}}\, \widetilde{\mathbb{S}}_\epsilon + \widetilde{\rho}_\epsilon \widetilde{\mathbf{f}}_\epsilon \quad \text{in } \widetilde{\Omega}_\epsilon \times (0,T), && (6.1.2) \end{aligned}$$

where $T > 0$, $\widetilde{\Omega}_\epsilon$ is defined in Section 3.5, $\widetilde{p}_\epsilon$ is the pressure, $\widetilde{\mathbb{S}}_\epsilon$ stands for the viscous stress tensor and $\widetilde{\mathbf{f}}_\epsilon$ represents the external forces (see [126]). We focus on isothermal and non-Newtonian fluids, which means that

$$\widetilde{\mathbb{S}}_\epsilon = \widetilde{P}(|\widetilde{D}\widetilde{\mathbf{u}}_\epsilon|)\widetilde{D}\widetilde{\mathbf{u}}_\epsilon, \qquad \widetilde{p}_\epsilon = c_p \widetilde{\rho}_\epsilon. \tag{6.1.3}$$

Without loss of generality, we put $c_p = 1$.

We focus on the rigorous derivation of the two-dimensional model from equations (6.1.1)–(6.1.2) under the Navier boundary conditions. To introduce the Navier boundary conditions we must first establish some notation. Symbols $\widetilde{\mathbf{n}}_\epsilon$ stand for unit outward normals to $\widetilde{\Omega}_\epsilon$. Similarly $\widetilde{\mathbf{t}}_\epsilon$ is any vector from the corresponding tangent plane. We denote parts of the boundary of domain $\widetilde{\Omega}_\epsilon$ as follows:

$$\widetilde{\Gamma}_{1,\epsilon} = \Theta_\epsilon\left(\Gamma_1\right),\ \widetilde{\Gamma}_{2,\epsilon} = \Theta_\epsilon\left(\Gamma_2\right), \tag{6.1.4}$$

where

$$\Gamma_1 = \partial S \times (0,1),\ \Gamma_2 = S \times \{0,1\} \tag{6.1.5}$$

(see (4.5.1) for the definition of Θ_ϵ). Using the notation, we prescribe the set of the Navier boundary conditions

$$\begin{aligned}
\widetilde{\mathbf{t}}_\epsilon \cdot \left(\widetilde{P}(|\widetilde{D}\widetilde{\mathbf{u}}_\epsilon|)\widetilde{D}\widetilde{\mathbf{u}}_\epsilon\widetilde{\mathbf{n}}_\epsilon\right) + q\widetilde{\mathbf{u}}_\epsilon \cdot \widetilde{\mathbf{t}}_\epsilon &= 0 \text{ on } \widetilde{\Gamma}_{1,\epsilon} \times (0,T), && (6.1.6)\\
\widetilde{\mathbf{t}}_\epsilon \cdot \left(\widetilde{P}(|\widetilde{D}\widetilde{\mathbf{u}}_\epsilon|)\widetilde{D}\widetilde{\mathbf{u}}_\epsilon\widetilde{\mathbf{n}}_\epsilon\right) + h(\epsilon)\widetilde{\mathbf{u}}_\epsilon \cdot \widetilde{\mathbf{t}}_\epsilon &= 0 \text{ on } \widetilde{\Gamma}_{2,\epsilon} \times (0,T), && (6.1.7)\\
\widetilde{\mathbf{u}}_\epsilon \cdot \widetilde{\mathbf{n}}_\epsilon &= 0 \text{ on } \partial\widetilde{\Omega}_\epsilon \times (0,T). && (6.1.8)
\end{aligned}$$

We suppose that $h(\epsilon) > 0$ behaves like $O(\epsilon)$ and $q > 0$. The asymptotic behavior of $h(\epsilon)$ will be discussed during the derivation of weak convergences of densities and velocity fields.

We further consider the initial conditions for the density and the momentum

$$\begin{aligned}
\widetilde{\rho}_\epsilon(\widetilde{x},0) &= \widetilde{\rho}_{0,\epsilon}(\widetilde{x}) \geq 0, && (6.1.9)\\
(\widetilde{\rho}_\epsilon\widetilde{\mathbf{u}}_\epsilon)(\widetilde{x},0) &= (\widetilde{\rho}_\epsilon\widetilde{\mathbf{u}}_\epsilon)_0(\widetilde{x}),\ \widetilde{x} \in \widetilde{\Omega}_\epsilon. && (6.1.10)
\end{aligned}$$

Hence, the variational formulation of our problem is

$$\int_0^T \int_{\widetilde{\Omega}_\epsilon} \left[\widetilde{\rho}_\epsilon \partial_t \widetilde{\varphi} + \widetilde{\rho}_\epsilon \widetilde{\mathbf{u}}_\epsilon \cdot \widetilde{\nabla}\widetilde{\varphi}\right] \, d\widetilde{x}dt = 0, \qquad (6.1.11)$$

$$\begin{aligned}
&\int_0^T \int_{\widetilde{\Omega}_\epsilon} \left[\widetilde{\rho}_\epsilon\widetilde{\mathbf{u}}_\epsilon \cdot \partial_t\widetilde{\psi} + \widetilde{\rho}_\epsilon\widetilde{\mathbf{u}}_\epsilon \otimes \widetilde{\mathbf{u}}_\epsilon : \widetilde{D}\widetilde{\psi} + \widetilde{\rho}_\epsilon \widetilde{\operatorname{div}}\ \widetilde{\psi}\right] \, d\widetilde{x}dt = \\
&= \int_0^T \int_{\widetilde{\Omega}_\epsilon} \left[\widetilde{P}(|\widetilde{D}\widetilde{\mathbf{u}}_\epsilon|)\widetilde{D}\widetilde{\mathbf{u}}_\epsilon : \widetilde{D}\widetilde{\psi} - \widetilde{\rho}_\epsilon\widetilde{\mathbf{f}}_\epsilon \cdot \widetilde{\psi}\right] \, d\widetilde{x}dt + \\
&+q\int_0^T \int_{\widetilde{\Gamma}_{1,\epsilon}} \widetilde{\mathbf{u}}_\epsilon \cdot \widetilde{\psi} \, d\widetilde{\Gamma}dt + h(\epsilon)\int_0^T \int_{\widetilde{\Gamma}_{2,\epsilon}} \widetilde{\mathbf{u}}_\epsilon \cdot \widetilde{\psi} \, d\widetilde{\Gamma}dt \qquad (6.1.12)
\end{aligned}$$

for any $\widetilde{\varphi} \in \mathcal{D}\left(\mathbb{R}^3 \times (0,T)\right)$ and $\widetilde{\psi} \in C_0^\infty(0,T; C^\infty(\overline{\widetilde{\Omega}_\epsilon})^3)$ satisfying the condition $\widetilde{\psi} \cdot \widetilde{\mathbf{n}}_\epsilon|_{\partial\widetilde{\Omega}_\epsilon \times (0,T)} = 0$.

Similarly as in [126], [127] and [212], we assume that function P satisfies, for any $\widetilde{U}, \widetilde{V} \in \widetilde{L}_M(\widetilde{\Omega}_\epsilon)^9$, the following conditions

$$\int_{\widetilde{\Omega}_\epsilon} P(|\widetilde{U}|)|\widetilde{U}|^2 \, d\widetilde{x} \geq \int_{\widetilde{\Omega}_\epsilon} M(|\widetilde{U}|) \, d\widetilde{x}, \qquad (6.1.13)$$

$$\int_{\widetilde{\Omega}_\epsilon} \left(P(|\widetilde{U}|)\widetilde{U} - P(|\widetilde{V}|)\widetilde{V}\right) : (\widetilde{U} - \widetilde{V}) \, d\widetilde{x} \geq 0, \qquad (6.1.14)$$

$$P(z)|z|^2 \text{ is a convex function for } z \geq 0, \qquad (6.1.15)$$

$$\int_{\widetilde{\Omega}_\epsilon} N(P(|\widetilde{U}|)|\widetilde{U}|) \, d\widetilde{x} \leq C\left(1 + \int_{\widetilde{\Omega}_\epsilon} M(|\widetilde{U}|)\right) \, d\widetilde{x}, \qquad (6.1.16)$$

$$P(|\widetilde{U} - \lambda\widetilde{V}|)(\widetilde{U} - \lambda\widetilde{V}) \xrightarrow{M} P(|\widetilde{U}|)\widetilde{U}, \text{ for } \lambda \to 0. \qquad (6.1.17)$$

For instance, the function

$$P(z) = \begin{cases} \frac{M(z)}{z^2}, & \text{for } z \neq 0, \\ 0, & \text{for } z = 0. \end{cases}$$

satisfies all of the conditions (6.1.13)–(6.1.17). The Orlicz spaces and Young functions used in above relations are defined in the next subsection.

6.1.1 Orlicz spaces

In this subsection, we give more details about the typical Orlicz function spaces for the kind of equations we introduced above.

Definition 6.1.1 *Let us define the Young functions $\Phi_\gamma(z) := (1+z)\ln^\gamma(1+z)$, $\gamma > 1$, and $\Phi_1(z) := z\ln(z+1)$. Functions Ψ_γ, $\gamma \geq 1$, denote the complementary functions to Φ_γ, $\gamma \geq 1$. Subsequently, we define $M(z) := \mathrm{e}^z - z - 1$ and its complementary function $N(z) = (1+z)\ln(1+z) - z$. Further, we denote $\Phi_{1/\alpha}(z)$, $\alpha \in (1, +\infty)$, the Young functions with growth $z\ln^{1/\alpha} z$, $z \geq z_0 > 0$. $\Psi_{1/\alpha}(z)$ are their complementary functions.*

It is apparent that

- $\Phi_\gamma(z) \sim O(z\ln^\gamma z)$, $\gamma > 0$, and $M(z) \sim O(\mathrm{e}^z)$,
- $\Psi_\gamma(z) \sim O(\mathrm{e}^{z^{1/\gamma}})$, $\gamma > 0$, and $N(z) \sim O(z\ln z)$,
- $L_{\Phi_1}(Q) = L_N(Q)$ and also $L_{\Psi_1}(Q) = L_M(Q)$,
- the Young functions Φ_γ, $\gamma \geq 1$, satisfy the Δ_2-condition,
- if $\gamma_2 > \gamma_1 \geq 1$, then $L_{\Phi_{\gamma_2}} \subset L_{\Phi_{\gamma_1}}$ and $L_{\Psi_{\gamma_1}} \subset L_{\Psi_{\gamma_2}}$,
- if $u \in L_{\Phi_\gamma}(Q)$, $\gamma \geq 1$, then $\int_Q \Phi_\gamma(|u(x)|)\, dx < +\infty$,
- if $u \in L_{\Psi_\gamma}(Q)$, $\gamma \geq 1$, then $\int_Q \Psi_{\gamma'}(|u(x)|)\, dx < +\infty$, $\forall \gamma' > \gamma$.

6.1.2 Main result

The main result of Section 5 is stated in the following theorem.

Theorem 6.1.2 *Let us assume that couples $\langle \rho_\epsilon, \mathbf{u}_\epsilon \rangle$, $\epsilon \in (0,1)$, satisfying*

$$\begin{aligned} \rho_\epsilon &\in L^\infty(0,T; L_{\Phi_\gamma}(\Omega)), \\ \mathbf{v}_\epsilon &\in L^p(0,T; W^{1,p}(\Omega)^3) \cap L^2(0,T; L^2(\partial\Omega)^3) \end{aligned}$$

with $\mathbf{v}_\epsilon := (\mathbf{u}_\epsilon \cdot \mathbf{g}_{1,\epsilon}, \mathbf{u}_\epsilon \cdot \mathbf{g}_{2,\epsilon}, \mathbf{u}_\epsilon \cdot \mathbf{g}_{3,\epsilon})$ for arbitrary but fixed $\gamma > 3$ and $p > 3$, are weak solutions to the transformed equations (6.2.5)–(6.2.6) with initial states $\rho_{0,\epsilon} \in L_{\Phi_\gamma}(\Omega)$ and $\frac{|(\rho_\epsilon \mathbf{u}_\epsilon)_0|^2}{2\rho_{0,\epsilon}}\sqrt{d_\epsilon} \in L^1(\Omega)$ satisfying (6.3.45)–(6.3.47). In addition, we assume that the Navier boundary conditions (6.1.6)–(6.1.8) hold and $\omega_\epsilon(\mathbf{u}_\epsilon) \in \tilde{L}_M(\Omega \times (0,T))^9$. Further, we suppose that function P complies with conditions (6.1.13)–(6.1.17), $\mathbf{f}_\epsilon \to \mathbf{f}$ in $L^\infty(\Omega \times (0,T))^3$ and $\mathbf{f}_\epsilon \cdot \mathbf{g}^{j,\epsilon} \in L^\infty(\Omega \times (0,T))^3$, $j = 1,2,3$, $h(\epsilon) > 0$ behaves like $O(\epsilon)$, $q > 0$ and covariant basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} \subset L^\infty(\Omega)^3$ satisfies conditions $\partial_\alpha \mathbf{a}_j$ and $\partial^2_{\alpha\beta}\mathbf{a}_3 \in L^\infty(\Omega)^3$, where α, $\beta = 1,2$ and $j = 1,2,3$. Then (passing to subsequences if necessary)

$$\begin{aligned} \rho_\epsilon &\overset{*}{\rightharpoonup} \rho && \text{in } L^\infty\left(0,T; L_{\Phi_\gamma}(\Omega)\right), \\ \rho_\epsilon &\to \rho && \text{in } C\left([0,T]; [WL_{\Psi_\gamma}(\Omega)]'\right), \\ \omega_\epsilon(\mathbf{u}_\epsilon) &\overset{N}{\rightharpoonup} \omega(\mathbf{u}) && \\ \mathbf{u}_\epsilon \cdot \mathbf{g}_{\alpha,\epsilon} &\rightharpoonup \mathbf{u} \cdot \mathbf{a}_\alpha && \text{in } L^p\left(0,T; W^{1,p}(\Omega)\right) \cap L^2\left(0,T; L^2(\partial\Omega)\right), \\ & && \alpha = 1,2, \\ \mathbf{u}_\epsilon \cdot \mathbf{a}_3 &\to 0 && \text{in } L_M(\Omega \times (0,T)). \end{aligned}$$

In addition, couple $\langle \hat{\rho}, \hat{\mathbf{u}} \rangle$, where $\hat{\rho} = \int_0^1 \rho \ \ dx_3$ and $\mathbf{u} = (\mathbf{u} \cdot \mathbf{a}_1)\mathbf{a}^1 + (\mathbf{u} \cdot \mathbf{a}_2)\mathbf{a}^2$, $\hat{\mathbf{u}} \cdot \hat{\mathbf{n}}|_{\partial S \times (0,T)} = 0$, is a weak solution to the equations (6.4.1)–(6.4.2) and complies with the energy equality (6.4.3).

6.2 Transformation

In this section, we show how the transformations on the referential domain Ω introduced in Section 3.5 change the variational formulations (6.1.11)–(6.1.12) together with respective energy equality.

6.2.1 Transformation of partial derivatives

For transformed velocity and density, we employ the notation

$$\begin{aligned} \mathbf{u}_\epsilon &: \Omega \times (0,T) \to \mathbb{R}^3, \\ \rho_\epsilon &: \Omega \times (0,T) \to \mathbb{R}, \end{aligned}$$

where $\mathbf{u}_\epsilon(x,t) := \widetilde{\mathbf{u}}_\epsilon(\Theta_\epsilon(x),t)$ and $\rho_\epsilon(x,t) := \widetilde{\rho}_\epsilon(\Theta_\epsilon(x),t)$, for all $(x,t) \in \Omega \times (0,T)$. We denote $\widetilde{x} = \Theta_\epsilon(x)$ and also $x = \Theta_\epsilon^{-1}(\widetilde{x})$. Thus, we can write $\mathbf{u}_\epsilon(x,t) = \widetilde{\mathbf{u}}_\epsilon(\widetilde{x},t)$ and $\rho_\epsilon(x,t) = \widetilde{\rho}_\epsilon(\widetilde{x},t)$.

We express the first spatial partial derivative of a scalar function $\widetilde{\varphi}$ according to the chain rule in the following way

$$\widetilde{\partial}_j \widetilde{\varphi}(\widetilde{x},t) = \widetilde{\partial}_j \varphi(\Theta_\epsilon^{-1}(\widetilde{x}),t) = \partial_l \varphi(x,t)[\mathbf{g}^{l,\epsilon}]_j.$$

Similarly, we derive the first spatial partial derivative of the vector function $\widetilde{\mathbf{u}}_\epsilon$ as follows

$$\widetilde{\partial}_j \widetilde{u}_{i,\epsilon}(\widetilde{x},t) = \widetilde{\partial}_j u_{i,\epsilon}(\Theta_\epsilon^{-1}(\widetilde{x}),t) = \partial_l u_{i,\epsilon}(x,t)[\mathbf{g}^{l,\epsilon}]_j = \partial_l \mathbf{u}_\epsilon(x,t) \cdot \mathbf{g}_{k,\epsilon}[\mathbf{g}^{k,\epsilon}]_i[\mathbf{g}^{l,\epsilon}]_j,$$

where the last equality follows from

$$\partial_l u_{i,\epsilon} = [\partial_l \mathbf{u}_\epsilon]_i = \partial_l \mathbf{u}_\epsilon \cdot \mathbf{g}_{k,\epsilon}[\mathbf{g}^{k,\epsilon}]_i$$

(see (4.5.6)).

Transformation of the symmetric part of the gradient can be performed in the following way

$$[\widetilde{D}\widetilde{\mathbf{u}}_\epsilon]_{ij} = [\bar{\omega}_\epsilon(\mathbf{u}_\epsilon)]_{lk}\,[\mathbf{r}^{k,\epsilon}]_i[\mathbf{r}^{l,\epsilon}]_j = \left[R_\epsilon^T \bar{\omega}_\epsilon(\mathbf{u}_\epsilon) R_\epsilon\right]_{ij} =: [\omega_\epsilon(\mathbf{u}_\epsilon)]_{ij}\,, \qquad (6.2.1)$$

where

$$\bar{\omega}_\epsilon(\mathbf{u}_\epsilon) := \begin{pmatrix} \partial_1 \mathbf{u}_\epsilon \cdot \mathbf{g}_{1,\epsilon} & \frac{1}{2}\left(\partial_1 \mathbf{u}_\epsilon \cdot \mathbf{g}_{2,\epsilon} + \partial_2 \mathbf{u}_\epsilon \cdot \mathbf{g}_{1,\epsilon}\right) & \frac{1}{2}\left(\frac{\partial_1 \mathbf{u}_\epsilon \cdot \epsilon \mathbf{a}_3 + \partial_3 \mathbf{u}_\epsilon \cdot \mathbf{g}_{1,\epsilon}}{\epsilon}\right) \\ \cdot & \partial_2 \mathbf{u}_\epsilon \cdot \mathbf{g}_{2,\epsilon} & \frac{1}{2}\left(\frac{\partial_2 \mathbf{u}_\epsilon \cdot \epsilon \mathbf{a}_3 + \partial_3 \mathbf{u}_\epsilon \cdot \mathbf{g}_{2,\epsilon}}{\epsilon}\right) \\ \text{sym} & \cdot & \frac{\partial_3 \mathbf{u}_\epsilon \cdot \epsilon \mathbf{a}_3}{\epsilon^2} \end{pmatrix} \qquad (6.2.2)$$

and R_ϵ is defined in (4.5.9).

It is easy to check that for any $p \geq 3$, there exist $r_1,\ r_2 > 0$ such that for all $\epsilon \in$ (0,1) the following relation holds

$$r_1 \|\bar{\omega}_\epsilon(\mathbf{u}_\epsilon)\|_p \leq \|\omega_\epsilon(\mathbf{u}_\epsilon)\|_p \leq r_2 \|\bar{\omega}_\epsilon(\mathbf{u}_\epsilon)\|_p, \qquad (6.2.3)$$

because R_ϵ is convergent for $\epsilon \to 0$ in $W^{1,\infty}(\Omega)^9$. Furthermore, R_ϵ does not tend to zero for $\epsilon \to 0$ due to the formula (4.5.17). In the following sections, we do not need only the equivalence of $\bar{\omega}_\epsilon(\mathbf{u}_\epsilon)$ and $\omega_\epsilon(\mathbf{u}_\epsilon)$ in the L^p-norm, but also in the L_M-norm. This equivalence can be proved similarly to the inequality (6.2.3).

The transformation of $\widetilde{\mathbf{u}}_\epsilon \cdot \widetilde{\nabla} \widetilde{\varphi}$ leads to

$$\widetilde{u}_{i,\epsilon} \widetilde{\partial}_i \widetilde{\varphi} = u_{i,\epsilon} \partial_l \varphi [\mathbf{g}^{l,\epsilon}]_i = \mathbf{u}_\epsilon^T R_\epsilon E_\epsilon \nabla \varphi$$

(see (4.5.8) for the definition of E_ϵ). The transformation of $\widetilde{\text{div}}\ \widetilde{\psi}$ is done similarly

$$\widetilde{\text{div}}\ \widetilde{\psi} = \widetilde{\partial}_i \widetilde{\psi}_i = \partial_l \psi_i [\mathbf{g}^{l,\epsilon}]_i = \nabla \psi : R_\epsilon E_\epsilon. \qquad (6.2.4)$$

We remark that term $\nabla \psi : R_\epsilon E_\epsilon$ is the trace of $\omega_\epsilon(\psi)$ because $\widetilde{\text{div}}\ \widetilde{\psi}$ is the trace of $\widetilde{D}\widetilde{\psi}$.

6.2.2 Transformation of the governing equations

According to [44], we use the following equalities

$$\begin{aligned} d\widetilde{x} &= \sqrt{g_\epsilon}\ dx = \epsilon\sqrt{d_\epsilon}\ dx, \\ d\widetilde{\Gamma} &= |R_\epsilon E_\epsilon \mathbf{n}|\sqrt{g_\epsilon}\ d\Gamma = |R_\epsilon E_\epsilon \mathbf{n}|\epsilon\sqrt{d_\epsilon}\ d\Gamma, \end{aligned}$$

to arrive at the transformed equations of the variational formulation (6.1.11)–(6.1.12). It holds that $\mathbf{n} = (n_1, n_2, 0)$ on Γ_1, $\mathbf{n} = (0, 0, \pm 1)$ on Γ_2. Therefore,

$$\begin{aligned} |R_\epsilon E_\epsilon \mathbf{n}| &= \sqrt{\sum_{i,j=1}^{2} n_i g^{ij,\epsilon} n_j} \text{ on } \Gamma_1, \\ |R_\epsilon E_\epsilon \mathbf{n}| &= \epsilon^{-1} \text{ on } \Gamma_2. \end{aligned}$$

Now, we can divide both equations by ϵ and arrive at the transformed variational formulation

$$\int_0^T \int_\Omega [\rho_\epsilon \partial_t \varphi + \rho_\epsilon \mathbf{u}_\epsilon^T R_\epsilon E_\epsilon \nabla\varphi]\sqrt{d_\epsilon}\ dxdt = 0, \tag{6.2.5}$$

$$\begin{aligned} &\int_0^T \int_\Omega [\rho_\epsilon \mathbf{u}_\epsilon \cdot \partial_t \psi + \rho_\epsilon \mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon : \omega_\epsilon(\psi) + \rho_\epsilon \nabla\psi : R_\epsilon E_\epsilon]\sqrt{d_\epsilon}\ dxdt = \\ &= \int_0^T \int_\Omega [P(|\omega_\epsilon(\mathbf{u}_\epsilon)|)\,\omega_\epsilon(\mathbf{u}_\epsilon) : \omega_\epsilon(\psi) - \rho_\epsilon \mathbf{f}_\epsilon \cdot \psi]\sqrt{d_\epsilon}\ dxdt + \\ &+ q\int_0^T \int_{\Gamma_1} \mathbf{u}_\epsilon \cdot \psi |R_\epsilon E_\epsilon \mathbf{n}|\sqrt{d_\epsilon}\ d\Gamma dt + \frac{h(\epsilon)}{\epsilon}\int_0^T \int_{\Gamma_2} \mathbf{u}_\epsilon \cdot \psi\sqrt{d_\epsilon}\ d\Gamma dt \end{aligned} \tag{6.2.6}$$

for any $\varphi \in \mathcal{D}\left(\mathbb{R}^3 \times (0,T)\right)$ and $\psi \in C_0^\infty(0,T; C^\infty(\bar{\Omega})^3)$, $\psi \cdot \mathbf{n}|_{\partial\Omega\times(0,T)} = 0$.

After imposing the same transformation as for the variational formulation to the renormalized continuity equation (see [118] or [127] for its original form), we get

$$\int_0^T \int_\Omega \left[b(\rho_\epsilon)\partial_t\varphi + b(\rho_\epsilon)\mathbf{u}_\epsilon^T R_\epsilon E_\epsilon \nabla\varphi\,(b(\rho_\epsilon) - \rho_\epsilon b'(\rho_\epsilon))\,\nabla\mathbf{u}_\epsilon : R_\epsilon E_\epsilon\right]\varphi\sqrt{d_\epsilon}\ dxdt = 0 \tag{6.2.7}$$

for any $\varphi \in \mathcal{D}\left(\mathbb{R}^3 \times (0,T)\right)$.

6.2.3 Energy equality and its transformation

For any $t \in [0,T]$, we have the energy equality expressed by the following formula (see [127])

$$\begin{aligned} &\int_{\widetilde{\Omega}_\epsilon} \frac{\widetilde{\rho}_\epsilon(t)\,|\widetilde{\mathbf{u}}_\epsilon(t)|^2}{2} + \widetilde{\rho}_\epsilon(t)\ln(\widetilde{\rho}_\epsilon(t))\ d\tilde{x} + \int_0^t \int_{\widetilde{\Omega}_\epsilon} P(|\widetilde{D}\widetilde{\mathbf{u}}_\epsilon|)|\widetilde{D}\widetilde{\mathbf{u}}_\epsilon|^2\ d\tilde{x}ds + \\ &+ q\int_0^t \int_{\widetilde{\Gamma}_{1,\epsilon}} |\widetilde{\mathbf{u}}_\epsilon|^2\ d\widetilde{\Gamma}ds + h(\epsilon)\int_0^t \int_{\widetilde{\Gamma}_{2,\epsilon}} |\widetilde{\mathbf{u}}_\epsilon|^2\ d\widetilde{\Gamma}ds = \\ &= \int_0^t \int_{\widetilde{\Omega}_\epsilon} \widetilde{\rho}_\epsilon \widetilde{\mathbf{f}}_\epsilon \cdot \widetilde{\mathbf{u}}_\epsilon\ d\widetilde{x}ds + \int_{\widetilde{\Omega}_\epsilon} \frac{|(\widetilde{\rho}_\epsilon \widetilde{\mathbf{u}}_\epsilon)_0|^2}{2\widetilde{\rho}_{\epsilon,0}} + \widetilde{\rho}_{\epsilon,0}\ln(\widetilde{\rho}_{\epsilon,0})\ d\tilde{x}. \end{aligned} \tag{6.2.8}$$

By transforming (6.2.8), we obtain

$$\int_\Omega \left[\frac{\rho_\epsilon(t)|\mathbf{u}_\epsilon(t)|^2}{2} + \rho_\epsilon(t)\ln(\rho_\epsilon(t))\right]\sqrt{d_\epsilon}\ dx + \int_0^t \int_\Omega P\left(|\omega_\epsilon(\mathbf{u}_\epsilon)|\right)|\omega_\epsilon(\mathbf{u}_\epsilon)|^2\sqrt{d_\epsilon}\ dxds +$$

$$+q\int_0^t\int_{\Gamma_1}|\mathbf{u}_\epsilon|^2|R_\epsilon E_\epsilon \mathbf{n}|\sqrt{d_\epsilon}\;d\Gamma ds+\frac{h(\epsilon)}{\epsilon}\int_0^t\int_{\Gamma_2}|\mathbf{u}_\epsilon|^2\sqrt{d_\epsilon}\;d\Gamma ds=$$

$$=\int_0^t\int_\Omega \rho_\epsilon\bar{\mathbf{f}}_\epsilon\cdot\mathbf{v}_\epsilon\sqrt{d_\epsilon}\;dxds+\int_\Omega\left[\frac{|(\rho_\epsilon\mathbf{u}_\epsilon)_0|^2}{2\rho_{\epsilon,0}}+\rho_{\epsilon,0}\ln(\rho_{\epsilon,0})\right]\sqrt{d_\epsilon}\;dx \qquad (6.2.9)$$

for any $t\in[0,T]$, where

$$\begin{aligned}\bar{\mathbf{f}}_\epsilon &:= (\mathbf{f}_\epsilon\cdot\mathbf{g}^{1,\epsilon},\mathbf{f}_\epsilon\cdot\mathbf{g}^{2,\epsilon},\mathbf{f}_\epsilon\cdot\mathbf{g}^{3,\epsilon}),\\ \mathbf{v}_\epsilon &:= (\mathbf{u}_\epsilon\cdot\mathbf{g}_{1,\epsilon},\mathbf{u}_\epsilon\cdot\mathbf{g}_{2,\epsilon},\mathbf{u}_\epsilon\cdot\mathbf{g}_{3,\epsilon}).\end{aligned}$$

It is obvious that

$$\begin{aligned}\bar{\mathbf{f}}_\epsilon\cdot\mathbf{v}_\epsilon &= \bar{f}_{i,\epsilon}v_{i,\epsilon}=(\mathbf{f}_\epsilon\cdot\mathbf{g}^{i,\epsilon})(\mathbf{u}_\epsilon\cdot\mathbf{g}_{i,\epsilon})=\\ &= (\mathbf{f}_\epsilon\cdot\mathbf{g}^{i,\epsilon})\mathbf{g}_{i,\epsilon}\cdot(\mathbf{u}_\epsilon\cdot\mathbf{g}_{j,\epsilon})\mathbf{g}^{j,\epsilon}=\mathbf{f}_\epsilon\cdot\mathbf{u}_\epsilon.\end{aligned}$$

We need to use $\bar{\mathbf{f}}_\epsilon\cdot\mathbf{v}_\epsilon$ instead of $\mathbf{f}_\epsilon\cdot\mathbf{u}_\epsilon$ for making a priori estimates (see inequality (6.3.5)), because a variant of Korn's inequality holds for $\mathbf{v}_\epsilon$ (see Theorem 6.3.1).

6.3 Proof of the limiting 2D equations

The first step of the proof concerns the variant of Korn's inequality. We need this inequality to perform a priori estimates in section 5.3.2 and subsequently show boundedness of $\{\rho_\epsilon\}_{\epsilon\in(0,1)}$ and $\{\mathbf{v}_\epsilon\}_{\epsilon\in(0,1)}$, and perform weak limits. In section 5.3.3, we pass to the limits in the equations (6.2.5)–(6.2.6). As the last step, we perform the limit passage also for the energy equality (6.2.9).

6.3.1 Korn's inequality

In this section, we prove the variant of the first Korn inequality for functions from $W^{1,p}(\Omega)^3$, $p>3$. This inequality is subsequently used to derive a priori estimates for ρ_ϵ and $\mathbf{u}_\epsilon$ in Section 6.3.2.

From [72], we know that

$$\|\mathbf{w}\|_{1,p}\le C\left(\|D\mathbf{w}\|_p+\|\mathbf{w}\|_p\right) \qquad (6.3.1)$$

holds for any $\mathbf{w}\in W^{1,p}(\Omega)^3$, $p\ge 2$. It is an easy consequence to prove that there exists constant $C(\Omega,p)>0$ such that

$$\|\mathbf{w}\|_{1,p}\le C(\Omega,p)\left(\|D\mathbf{w}\|_p+\|\mathbf{w}\|_{2,\Gamma}\right). \qquad (6.3.2)$$

Without the loss of generality, we denote $\mathbf{u}_\epsilon=\mathbf{u}_\epsilon(t)$ in the following theorem. Variable $t\in[0,T]$ is arbitrary but fixed.

Theorem 6.3.1 *Let $\mathbf{u}_\epsilon\in W^{1,p}(\Omega)^3$, $p>3$, be such that $\mathbf{u}_\epsilon\cdot\mathbf{n}=\mathbf{u}_\epsilon\cdot\mathbf{a}_3=0$ on $\Gamma:=S\times\{0\}$. We define $\mathbf{v}_\epsilon:=(\mathbf{u}_\epsilon\cdot\mathbf{g}_{1,\epsilon},\mathbf{u}_\epsilon\cdot\mathbf{g}_{2,\epsilon},\mathbf{u}_\epsilon\cdot\epsilon\mathbf{a}_3)$. Then there exists $C=C(\Omega,p)>0$, such that*

$$\|\mathbf{v}_\epsilon\|_{1,p}\le C\left(\|\bar{\omega}_\epsilon(\mathbf{u}_\epsilon)\|_p+\|\mathbf{u}_\epsilon\|_{2,\Gamma}\right),\quad \forall\epsilon>0, \qquad (6.3.3)$$

where $\bar{\omega}_\epsilon(\mathbf{u}_\epsilon)$ is defined by (6.2.2).

Proof: Assume the contrary: without loss of generality, there exists a sequence $\{\mathbf{v}_{\epsilon_n}\}_{n=1}^{+\infty}$ generated by $\{\mathbf{u}_{\epsilon_n}\}_{n=1}^{+\infty}$, where $\epsilon_n\to 0$ as n approaches infinity, such that $\|\mathbf{v}_{\epsilon_n}\|_{1,p}=1$ and

$$\frac{1}{n}\ge\|\bar{\omega}_{\epsilon_n}(\mathbf{u}_{\epsilon_n})\|_p+\|\mathbf{u}_{\epsilon_n}\|_{2,\Gamma}.$$

Hence,

$$\mathbf{u}_{\epsilon_n} \to 0 \text{ in } L^2(\Gamma)^3, \quad \bar{\omega}_{\epsilon_n}(\mathbf{u}_{\epsilon_n}) \to 0 \text{ in } L^p(\Omega)^9. \tag{6.3.4}$$

In addition, from the definition of $\mathbf{v}_{\epsilon_n}$, it follows that $v_{3,\epsilon_n} = 0$ on Γ. From boundedness of sequence $\{\mathbf{v}_{\epsilon_n}\}_{n=1}^{+\infty}$ and imbedding $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow C(\bar{\Omega})$, we get the convergences (passing to a subsequence if necessary)

$$\begin{aligned} \mathbf{v}_{\epsilon_n} &\rightharpoonup \mathbf{v} \quad \text{in } W^{1,p}(\Omega)^3, \\ \mathbf{v}_{\epsilon_n} &\to \mathbf{v} \quad \text{in } C(\bar{\Omega})^3. \end{aligned}$$

We will arrive at a contradiction in three steps:

1. *We prove that* $\{D\mathbf{v}_{\epsilon_n}\}_{n=1}^{+\infty}$ *is convergent in* $L^p(\Omega)^9$.

Let us analyze the terms of $\bar{\omega}_{\epsilon_n}(\mathbf{u}_{\epsilon_n})$ one by one. We know that

$$\|\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{a}_3\|_p \le \frac{\epsilon_n}{n}.$$

Hence, $\partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{a}_3 = \partial_3(\mathbf{u}_{\epsilon_n} \cdot \mathbf{a}_3) \to \partial_3(\mathbf{u} \cdot \mathbf{a}_3) = 0$ in $L^p(\Omega)$. However, $(\mathbf{u} \cdot \mathbf{a}_3)(x_1, x_2, 0) = 0$ for all $(x_1,x_2) \in S$. Thus, $v_3 = \mathbf{u} \cdot \mathbf{a}_3 = 0$ in Ω.

Next, $[\bar{\omega}_{\epsilon_n}(\mathbf{u}_{\epsilon_n})]_{11}$ can be written as

$$\partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{g}_{1,\epsilon_n} = \partial_1(\mathbf{u}_{\epsilon_n} \cdot \mathbf{g}_{1,\epsilon_n}) - \mathbf{u}_{\epsilon_n} \cdot \partial_1 \mathbf{g}_{1,\epsilon_n}.$$

From the definition of $\mathbf{g}_{1,\epsilon_n}$ (4.5.2), it follows that $\partial_1 \mathbf{g}_{1,\epsilon_n} = \partial_1 \mathbf{a}_1 + \epsilon_n x_3 \partial_{11}^2 \mathbf{a}_3$. Therefore, $\partial_1 \mathbf{g}_{1,\epsilon_n} \in L^\infty(\Omega)^3$ can be written as

$$\partial_1 \mathbf{g}_{1,\epsilon_n} = c_{1,\epsilon_n} \mathbf{g}_{1,\epsilon_n} + c_{2,\epsilon_n} \mathbf{g}_{2,\epsilon_n} + c_{3,\epsilon_n} \mathbf{a}_3,$$

where $c_{\alpha,\epsilon_n} = \partial_1 \mathbf{g}_{1,\epsilon_n} \cdot \mathbf{g}^{\alpha,\epsilon_n} \to c_\alpha$ in $L^\infty(\Omega)$, $\alpha = 1,2$, and $c_{3,\epsilon_n} = \partial_1 \mathbf{g}_{1,\epsilon_n} \cdot \mathbf{a}_3 \to c_3$ in $L^\infty(\Omega)$ due to convergences (4.5.11), (4.5.15) and (4.5.16). Hence, $\mathbf{u}_{\epsilon_n} \cdot \partial_1 \mathbf{g}_{1,\epsilon_n} \to c_1 v_1 + c_2 v_2$ in $L^\infty(\Omega)$. Together with the convergence $\partial_1 \mathbf{u}_{\epsilon_n} \cdot \mathbf{g}_{1,\epsilon_n} \to 0$ in $L^p(\Omega)$, we get

$$\partial_1(\mathbf{u}_{\epsilon_n} \cdot \mathbf{g}_{1,\epsilon_n}) = \partial_1 v_{1,\epsilon_n} \to c_1 v_1 + c_2 v_2 \quad \text{in } L^p(\Omega).$$

Similarly, we show that also the remaining terms of $D\mathbf{v}_{\epsilon_n}$ converge in $L^p(\Omega)$.

2. *We show that* $\{\mathbf{v}_{\epsilon_n}\}_{n=1}^{+\infty}$ *is convergent in* $W^{1,p}(\Omega)^3$.

We use Korn's inequality (6.3.2) for function $\mathbf{w} \in W^{1,p}(\Omega)^3$. We already know that $\mathbf{u}_{\epsilon_n} \to 0$ in $L^2(\Gamma)^3$. Hence, also $\mathbf{v}_{\epsilon_n} \to 0$ in $L^2(\Gamma)^3$. Together with the convergence of $D\mathbf{v}_{\epsilon_n}$ we get

$$\|\mathbf{v}_{\epsilon_n} - \mathbf{v}_{\epsilon_m}\|_{1,p} \le \bar{C}(\Omega, p) \left(\|D\mathbf{v}_{\epsilon_n} - D\mathbf{v}_{\epsilon_m}\|_p + \|\mathbf{v}_{\epsilon_n} - \mathbf{v}_{\epsilon_m}\|_{2,\Gamma} \right),$$

which implies the convergence of $\{\mathbf{v}_{\epsilon_n}\}_{n=1}^{+\infty}$ in $W^{1,p}(\Omega)^3$.

3. *To arrive at a contradiction, we prove that* $\|\mathbf{v}\|_{1,p} = 1$ *and simultaneously* $\mathbf{v} = 0$.

From $\mathbf{v}_{\epsilon_n} \to \mathbf{v}$ in $W^{1,p}(\Omega)^3$ and $\|\mathbf{v}_{\epsilon_n}\|_{1,p} = 1$, it stems that $\|\mathbf{v}\|_{1,p} = 1$. According to the definition of $\mathbf{g}_{\alpha,\epsilon_n}$, $\alpha = 1, 2$, (4.5.2)–(4.5.3) we know that $\partial_3 \mathbf{g}_{\alpha,\epsilon_n} = \epsilon_n \partial_\alpha \mathbf{a}_3$. We can write

$$\partial_1 \mathbf{u}_{\epsilon_n} \cdot \epsilon_n \mathbf{a}_3 + \partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{g}_{1,\epsilon_n} = \partial_1 v_{3,\epsilon_n} + \partial_3 v_{1,\epsilon_n} - 2\epsilon_n \mathbf{u}_{\epsilon_n} \cdot \partial_1 \mathbf{a}_3.$$

It holds that $\epsilon_n \mathbf{u}_{\epsilon_n} \cdot \partial_1 \mathbf{a}_3 = \epsilon_n (d_{1,\epsilon_n} \mathbf{u}_{\epsilon_n} \cdot \mathbf{g}_{1,\epsilon_n} + d_{2,\epsilon_n} \mathbf{u}_{\epsilon_n} \cdot \mathbf{g}_{2,\epsilon_n}) \to 0$ in $L^\infty(\Omega)$, where $d_{\alpha,\epsilon_n} = \partial_1 \mathbf{a}_3 \cdot \mathbf{g}^{\alpha,\epsilon_n} \to d_\alpha$ in $L^\infty(\Omega)$, α = 1,2, due to convergences (4.5.15), (4.5.16), and the second step of this proof.

Due to $\partial_1 \mathbf{u}_{\epsilon_n} \cdot \epsilon_n \mathbf{a}_3 + \partial_3 \mathbf{u}_{\epsilon_n} \cdot \mathbf{g}_{1,\epsilon_n} \to 0$ and $\epsilon_n \mathbf{u}_{\epsilon_n} \cdot \partial_1 \mathbf{a}_3 \to 0$ in $L^p(\Omega)$, also $\partial_1 v_{3,\epsilon_n} + \partial_3 v_{1,\epsilon_n} \to 0$ in $L^p(\Omega)$. In addition,

$$\int_\Omega \partial_1 v_{3,\epsilon_n} \varphi \, dx = -\int_\Omega \epsilon_n (\mathbf{u}_{\epsilon_n} \cdot \mathbf{a}_3) \partial_1 \varphi \, dx \to 0,$$

where $\varphi \in \mathcal{D}(\Omega)$. Hence, both $\partial_1 v_{3,\epsilon_n} \to 0$ and $\partial_3 v_{1,\epsilon_n} \to 0$ in $\mathcal{D}'(\Omega)$. In addition with respect to the results of the second step of this proof, we have $\partial_1 v_{3,\epsilon_n} \rightharpoonup 0$ and $\partial_3 v_{1,\epsilon_n} \rightharpoonup 0$ in $L^p(\Omega)$. Therefore, $\partial_3 v_1 = 0$ almost everywhere. Similarly, we can show that $\partial_3 v_2 = 0$ almost everywhere. However, relation (6.3.4) gives us $v_i(x_1, x_2, 0) = 0$, $i = 1,2$, for all $(x_1,x_2) \in S$, which, together with $\partial_3 v_i = 0$, gives us $v_i = 0$ in Ω.

Let us remind you that in the first part of this proof, we have already shown that $v_3 = 0$. To sum it up, $\mathbf{v} = 0$ in Ω and we arrive at a contradiction. □

6.3.2 Boundedness and weak limits

First, we make prior estimates. Equation (6.2.5) implies the conservation of mass, i.e.

$$\int_\Omega \rho_\epsilon(t) \sqrt{d_\epsilon} \, dx = \int_\Omega \rho_{0,\epsilon} \sqrt{d_\epsilon} \, dx, \quad \forall t \in (0,T).$$

Therefore due to assumptions of Theorem 6.1.2 on $\bar{\mathbf{f}}_\epsilon$, the first integral on the right-hand side of the energy equality (6.2.9) can be estimated as follows

$$\begin{aligned} \left| \int_0^t \int_\Omega \rho_\epsilon \bar{\mathbf{f}}_\epsilon \cdot \mathbf{v}_\epsilon \sqrt{d_\epsilon} \, dxds \right| &\leq \int_0^t \|\mathbf{v}_\epsilon(s)\|_\infty \|\bar{\mathbf{f}}_\epsilon(s)\|_\infty \int_\Omega \rho_\epsilon(s) \sqrt{d_\epsilon} \, dxds \leq \\ &\leq C(\rho_{0,\epsilon}, \bar{\mathbf{f}}_\epsilon) \int_0^t \|\mathbf{v}_\epsilon(s)\|_{1,p} \, ds. \end{aligned}$$

In view of Young's inequality (4.1.3), inequalities (6.1.13) and (6.3.3), and estimate (6.2.3), we arrive at

$$\begin{aligned} \left| \int_0^t \int_\Omega \rho_\epsilon \bar{\mathbf{f}}_\epsilon \cdot \mathbf{v}_\epsilon \;\; dxds \right| &\leq C \left(C_1 \int_0^t \int_\Omega P\left(|\omega_\epsilon(\mathbf{u}_\epsilon)|\right) |\omega_\epsilon(\mathbf{u}_\epsilon)|^2 \, dxds + \right. \\ &\left. + C_1 \int_0^t \int_{S \times \{0\}} |\mathbf{u}_\epsilon|^2 \, dSds + C_2(C_1) \right), \end{aligned} \tag{6.3.5}$$

where $C_1 > 0$ can be made arbitrarily small.

From (6.2.9) and (6.3.5), we obtain the boundedness of

$$\{\sqrt{\rho_\epsilon}\, |\mathbf{u}_\epsilon|\}_{\epsilon \in (0,1)} \quad \text{in} \quad L^\infty\left(0,T; L^2(\Omega)\right), \tag{6.3.6}$$

$$\{\rho_\epsilon\}_{\epsilon \in (0,1)} \quad \text{in} \quad L^\infty\left(0,T; L_{\Phi_1}(\Omega)\right), \tag{6.3.7}$$

$$\{\omega_\epsilon(\mathbf{u}_\epsilon)\}_{\epsilon \in (0,1)} \quad \text{in} \quad \tilde{L}_M\left(\Omega \times (0,T)\right)^9, \tag{6.3.8}$$

$$\{\mathbf{v}_\epsilon\}_{\epsilon \in (0,1)} \quad \text{in} \quad L^p(0,T; W^{1,p}(\Omega)^3) \cap L^2(0,T; L^2(\partial\Omega)^3) \tag{6.3.9}$$

for any $p > 3$. From (6.3.9), we get the boundedness of

$$\{\mathbf{u}_\epsilon \cdot \mathbf{g}_{\alpha,\epsilon}\}_{\epsilon \in (0,1)} \text{ in } L^p\left(0,T; W^{1,p}(\Omega)\right) \cap L^2\left(0,T; L^2(\partial\Omega)\right), \; \alpha = 1,2. \tag{6.3.10}$$

However, we do not have any information on the boundedness of $\{\mathbf{u}_\epsilon \cdot \mathbf{a}_3\}_{\epsilon \in (0,1)}$ yet. Therefore, we prove that

$$\mathbf{u}_\epsilon \cdot \mathbf{a}_3 \to \mathbf{u} \cdot \mathbf{a}_3 = 0 \text{ in } L_M(\Omega \times (0,T)). \tag{6.3.11}$$

Due to (6.3.8), we have the boundedness of $\epsilon^{-1}\partial_3\mathbf{u}_\epsilon\cdot\mathbf{a}_3$ in $\widetilde{L}_M(\Omega\times(0,T))$. It means that $\partial_3\mathbf{u}_\epsilon\cdot\mathbf{a}_3=\partial_3(\mathbf{u}_\epsilon\cdot\mathbf{a}_3)\to 0$. In addition, it holds that

$$|(\mathbf{u}_\epsilon\cdot\mathbf{a}_3)(x_1,x_2,x_3)-(\mathbf{u}_\epsilon\cdot\mathbf{a}_3)(x_1,x_2,0)|=\left|\int_0^{x_3}\partial_3(\mathbf{u}_\epsilon\cdot\mathbf{a}_3)(x_1,x_2,y)\;dy\right|.$$

According to the boundary conditions, we have $(\mathbf{u}_\epsilon\cdot\mathbf{a}_3)(x_1,x_2,0)=0$. Thus,

$$|(\mathbf{u}_\epsilon\cdot\mathbf{a}_3)|\le\int_0^1|\partial_3(\mathbf{u}_\epsilon\cdot\mathbf{a}_3)|\;dx_3.$$

Multiplying this inequality by ϵ^{-1} and applying norm $\|\cdot\|_{L_M(\Omega\times(0,T))}$ lead to

$$\begin{aligned}\|\epsilon^{-1}(\mathbf{u}_\epsilon\cdot\mathbf{a}_3)\|_{L_M(\Omega\times(0,T))} &\le \|\epsilon^{-1}\partial_3(\mathbf{u}_\epsilon\cdot\mathbf{a}_3)\|_{L_M(\Omega\times(0,T))}\le\\ &\le C_1\int_0^T\int_\Omega M(|[\omega_\epsilon(\mathbf{u}_\epsilon)]_{33}|)\;dxdt+C_2.\end{aligned}$$

Hence, we arrive at the boundedness of sequence

$$\{\epsilon^{-1}(\mathbf{u}_\epsilon\cdot\mathbf{a}_3)\}_{\epsilon\in(0,1)}\text{ in }L_M(\Omega\times(0,T)).\tag{6.3.12}$$

Therefore, the convergence (6.3.11) holds true.

The boundedness of $\{\rho_\epsilon\}_{\epsilon\in(0,1)}$ in $L^\infty(0,T;L_{\Phi_1}(\Omega))$ can be extended to the space $L^\infty(0,T;L_{\Phi_\gamma}(\Omega))$. We remind that $\gamma>3$ (see Theorem 6.1.2). We proceed in the following way. First, we test the equation (6.2.7) by function $\varphi=\varphi(t)\in C_0^\infty(0,T)$ with $b(z)=\Phi_\gamma(z)$. We arrive at

$$\int_0^T\int_\Omega\Phi_\gamma(\rho_\epsilon)\varphi'(t)+\left[\left(\Phi_\gamma(\rho_\epsilon)-\rho_\epsilon\Phi'_\gamma(\rho_\epsilon)\right)\nabla\mathbf{u}_\epsilon:R_\epsilon E_\epsilon\right]\varphi(t)\sqrt{d_\epsilon}\;dxdt=0.\tag{6.3.13}$$

Function $\Phi_\gamma(z)-z\Phi'_\gamma(z)$ behaves assymptotically as $\Phi_{\gamma-1}(z)$. Furthermore, there exists a positive constant C such that $\Phi_1(\Phi_{\gamma-1}(z))\le C(\Phi_\gamma(z)+1)$ for $z\ge0$ (see [212]). Due to equivalence of the Young functions M and Ψ_1, relations (6.1.13), (6.3.8), and Young's inequality, we deduce the estimate

$$\begin{aligned}&\left|\int_0^T\int_\Omega\left[\left(\Phi_\gamma(\rho_\epsilon)-\rho_\epsilon\Phi'_\gamma(\rho_\epsilon)\right)\nabla\mathbf{u}_\epsilon:R_\epsilon E_\epsilon\right]\sqrt{d_\epsilon}\;dxdt\right|\le\\ &\le C(T)\left(\int_0^T\int_\Omega\left[\Phi_\gamma(\rho_\epsilon)+P\left(|\omega_\epsilon(\mathbf{u}_\epsilon)|\right)|\omega_\epsilon(\mathbf{u}_\epsilon)|^2\right]\sqrt{d_\epsilon}\;dxdt+1\right)\end{aligned}\tag{6.3.14}$$

where $C(T)>0$. With respect to (6.3.13), (6.3.14), (6.3.46), and Gronwall's lemma, we obtain the boundedness of

$$\{\rho_\epsilon\}_{\epsilon\in(0,1)}\;\text{ in }L^\infty(0,T;\tilde{L}_{\Phi_\gamma}(\Omega)).\tag{6.3.15}$$

We focus now on the boundedness of $\{\partial_t\rho_\epsilon\}_{\epsilon\in(0,1)}$ in the next step. Let us test equation (6.2.5) by function $\varphi(x,t)=\varphi_1(t)\psi(x)$, where $\varphi_1\in L^{p'}(0,T)$, $1/p+1/p'=1$, $p>3$, and $\psi\in W^1L_{\Psi_{\gamma-1}}(\Omega)$, $\gamma>3$. We can write

$$\begin{aligned}&\left|\int_0^T\varphi'_1\int_\Omega\rho_\epsilon\psi\sqrt{d_\epsilon}\;dxdt\right|=\left|\int_0^T\varphi_1\int_\Omega\rho_\epsilon\mathbf{u}_\epsilon^TR_\epsilon E_\epsilon\nabla\psi\sqrt{d_\epsilon}\;dxdt\right|=\\ &=\left|\int_0^T\varphi_1\int_\Omega\rho_\epsilon\Big[\left((\mathbf{u}_\epsilon\cdot\mathbf{g}_{1,\epsilon})\mathbf{g}^{1,\epsilon}+(\mathbf{u}_\epsilon\cdot\mathbf{g}_{2,\epsilon})\mathbf{g}^{2,\epsilon}\right)\left(\mathbf{g}^{1,\epsilon},\mathbf{g}^{2,\epsilon}\right)\hat\nabla\psi+\right.\\ &\left.+\epsilon^{-1}\mathbf{u}_\epsilon\cdot\mathbf{a}_3\partial_3\psi\Big]\sqrt{d_\epsilon}\;dxdt\right|,\end{aligned}\tag{6.3.16}$$

where $\hat{\nabla}\psi = (\partial_1\psi, \partial_2\psi)^T$ and $\left(\mathbf{g}^{1,\epsilon}, \mathbf{g}^{2,\epsilon}\right)$ is the 3×2 submatrix of the matrix R_ϵ. It is sufficient to estimate only the last term on the right-hand side of (6.3.16), because it is "the worst term". Due to (6.3.8), (6.3.10), (6.3.12), and (6.3.15), we get the boundedness of

$$\{\partial_t\rho_\epsilon\}_{\epsilon\in(0,1)} \text{ in } L^{p'}\left(0,T;[W^1L_{\Psi_{\gamma-1}}(\Omega)]'\right). \tag{6.3.17}$$

By the use of (4.5.11), (6.3.7)–(6.3.9), (6.3.15), (6.3.17), and theorem on compact imbedding (see [188], Lemma 9), we get (passing to subsequences if necessary)

$$\rho_\epsilon \overset{*}{\rightharpoonup} \rho \qquad \text{in } L^\infty\left(0,T;L_{\Phi_\gamma}(\Omega)\right), \tag{6.3.18}$$

$$\rho_\epsilon \to \rho \qquad \text{in } C\left([0,T];[W^1L_{\Psi_\gamma}(\Omega)]'\right), \tag{6.3.19}$$

$$\omega_\epsilon(\mathbf{u}_\epsilon) \overset{N}{\rightharpoonup} \zeta \tag{6.3.20}$$

$$\mathbf{u}_\epsilon\cdot\mathbf{g}_{\alpha,\epsilon} \rightharpoonup \mathbf{u}\cdot\mathbf{a}_\alpha \qquad \text{in } L^p\left(0,T;W^{1,p}(\Omega)\right)\cap L^2\left(0,T;L^2(\partial\Omega)\right), \quad \alpha=1,2. \tag{6.3.21}$$

We remind that for the third projection of $\mathbf{u}_\epsilon$ into the covariant basis, we have $\mathbf{u}_\epsilon\cdot\mathbf{a}_3 \to \mathbf{u}\cdot\mathbf{a}_3 = 0$ in $L_M(\Omega\times(0,T))$ – see (6.3.12).

From the definition of $\omega_\epsilon(\mathbf{u}_\epsilon)$ (6.2.1) and (6.2.2), we can see that

$$\zeta = R^T\begin{pmatrix} \partial_1\mathbf{u}\cdot\mathbf{a}_1 & \frac{1}{2}\left(\partial_1\mathbf{u}\cdot\mathbf{a}_2+\partial_2\mathbf{u}\cdot\mathbf{a}_1\right) & \zeta_{13}\\ \cdot & \partial_2\mathbf{u}\cdot\mathbf{a}_2 & \zeta_{23}\\ \text{sym} & \cdot & \zeta_{33}\end{pmatrix}R. \tag{6.3.22}$$

We prove that the limiting function $\mathbf{u}$ does not depend on the third spatial variable. From (4.5.2) and (6.3.8), we know that $\{\epsilon^{-1}\left(\partial_1\mathbf{u}_\epsilon\cdot\epsilon\mathbf{a}_3+\partial_3\mathbf{u}_\epsilon\cdot\mathbf{g}_{1,\epsilon}\right)\}_{\epsilon\in(0,1)}$ is bounded in $\widetilde{L}_M(\Omega\times(0,T))$. It holds that

$$\epsilon^{-1}\left(\partial_1\mathbf{u}_\epsilon\cdot\epsilon\mathbf{a}_3+\partial_3\mathbf{u}_\epsilon\cdot\mathbf{g}_{1,\epsilon}\right) = \partial_1(\mathbf{u}_\epsilon\cdot\mathbf{a}_3)+\epsilon^{-1}\partial_3(\mathbf{u}_\epsilon\cdot\mathbf{g}_{1,\epsilon}) - 2\mathbf{u}_\epsilon\cdot\partial_1\mathbf{a}_3.$$

After multiplying this equation by ϵ and by a test function $\varphi\in\mathcal{D}(\mathbb{R}^3)$, and integrating over Ω, we get

$$\int_\Omega \partial_3(\mathbf{u}_\epsilon\cdot\mathbf{g}_{1,\epsilon})\varphi\,dx = \epsilon\int_\Omega \epsilon^{-1}\left(\partial_1\mathbf{u}_\epsilon\cdot\epsilon\mathbf{a}_3+\partial_3\mathbf{u}_\epsilon\cdot\mathbf{g}_{1,\epsilon}\right)\varphi\,dx + +\epsilon\int_\Omega\left(2\mathbf{u}_\epsilon\cdot\partial_1\mathbf{a}_3-\partial_1(\mathbf{u}_\epsilon\cdot\mathbf{a}_3)\right)\varphi\,dx. \tag{6.3.23}$$

With respect to (6.3.8), (6.3.10), and (6.3.11), the right-hand side of equality (6.3.23) tends to zero for $\epsilon\to 0$. Finally, we have $\partial_3(\mathbf{u}\cdot\mathbf{a}_1)=0$ almost everywhere. Similarly, we can conclude that $\partial_3(\mathbf{u}\cdot\mathbf{a}_2)=0$ almost everywhere. In summary and together with (6.3.11), we arrive at

$$\partial_3\mathbf{u}=0, \tag{6.3.24}$$

almost everywhere, which means that $\mathbf{u}$ is independent of x_3.

Now, we pay our attention to convergences of nonlinear terms in equation (6.2.6). The convergences (passing to subsequences if necessary)

$$\rho_\epsilon(\mathbf{u}_\epsilon\cdot\mathbf{g}_{\alpha,\epsilon}) \overset{\Psi_\gamma}{\rightharpoonup} \rho(\mathbf{u}\cdot\mathbf{a}_\alpha),\ \alpha=1,2, \tag{6.3.25}$$

$$\rho_\epsilon(\mathbf{u}_\epsilon\cdot\mathbf{a}_3) \to \rho(\mathbf{u}\cdot\mathbf{a}_3)=0,\ \text{in } L_{\Phi_{\gamma-1}}(\Omega\times(0,T)), \tag{6.3.26}$$

where $\gamma>3$ (see Theorem 6.1.2), are immediate consequences of (6.3.11), (6.3.17), (6.3.21), and theorem concerning compact imbedding (see [188], Lemma 9). For instance, we prove the convergence (6.3.26) that differs from convergences (6.3.25). According to Hölder's inequality, it holds that

$$\|\rho_\epsilon(\mathbf{u}_\epsilon\cdot\mathbf{a}_3)\|_{L_{\Phi_{\gamma-1}}(\Omega\times(0,T))} = \sup_\varphi \int_0^T\int_\Omega |\rho_\epsilon(\mathbf{u}_\epsilon\cdot\mathbf{a}_3)\varphi|\;dxdt \le$$
$$\le C\|\mathbf{u}_\epsilon\cdot\mathbf{a}_3\|_{L_M(\Omega\times(0,T))}\|\rho_\epsilon\varphi\|_{L_N(\Omega\times(0,T))},$$

where the supremum is taken over all functions $\varphi\in\widetilde{L}_{\Psi_{\gamma-1}}(\Omega\times(0,T))$ such that

$$\int_0^T\int_\Omega \Psi_{\gamma-1}(|\varphi|)\;dxdt \le 1.$$

From (6.3.11), we know that $\|\mathbf{u}_\epsilon\cdot\mathbf{a}_3\|_{L_M(\Omega\times(0,T))}\to 0$. Therefore, it is sufficient to show the boundedness of $\|\rho_\epsilon\varphi\|_{L_N(\Omega\times(0,T))}$ for proving (6.3.26). The equivalence of Orlicz spaces L_N and L_{Φ_1}, and Young's inequality give us

$$\|\rho_\epsilon\varphi\|_{L_N(\Omega\times(0,T))} \le \int_0^T\int_\Omega \Phi_1(\rho_\epsilon|\varphi|)\;dxdt + C \le$$
$$\le \int_0^T\int_\Omega \rho_\epsilon\Phi_1(|\varphi|)\;dxdt + \int_0^T\int_\Omega |\varphi|\Phi_1(\rho_\epsilon)\;dxdt + C. \tag{6.3.27}$$

The second integral on the right-hand side of (6.3.27) is "the worst" and it is less or equal than

$$\int_0^T\int_\Omega \Psi_{\gamma-1}(|\varphi|)\;dxdt + \int_0^T\int_\Omega \Phi_{\gamma-1}(\Phi_1(\rho_\epsilon))\;dxdt \le$$
$$\le \int_0^T\int_\Omega \Psi_{\gamma-1}(|\varphi|)\;dxdt + C\int_0^T\int_\Omega \Phi_\gamma(\rho_\epsilon)\;dxdt.$$

Hence, we conclude that convergence (6.3.26) holds true.

To overcome the second term on the left-hand side of equation (6.2.6), we consider "the worst integrals" in (6.2.6) and prove their boundedness. First, we show that from (6.3.10), (6.3.11) and (6.3.15) it follows the boundedness of

$$\int_0^T\int_\Omega \rho_\epsilon\mathbf{u}_\epsilon\otimes\mathbf{u}_\epsilon : \omega_\epsilon(\psi)\sqrt{d_\epsilon}\;dx\;dt \tag{6.3.28}$$

for any $\epsilon\in(0,1)$ and $\psi\in L^q(0,T;W^1E_{\Psi_{\gamma-2}}(\Omega)^3)$, where $2/p+1/q=1$, $\gamma>3$ (see Theorem 6.1.2), and $\psi\cdot\mathbf{n}|_{\partial\Omega\times(0,T)}=0$. Let us use formulas (4.5.9), (6.2.1), and (6.2.2), and perform the following reasoning:

$$\mathbf{u}_\epsilon\otimes\mathbf{u}_\epsilon : \omega_\epsilon(\psi) = u_{i,\epsilon}u_{j,\epsilon}\,[\bar\omega_\epsilon(\psi)]_{lk}\,[\mathbf{r}^{k,\epsilon}]_i[\mathbf{r}^{l,\epsilon}]_j =$$
$$= (\mathbf{u}_\epsilon\cdot\mathbf{r}^{k,\epsilon})(\mathbf{u}_\epsilon\cdot\mathbf{r}^{l,\epsilon})\,[\bar\omega_\epsilon(\psi)]_{lk}\,. \tag{6.3.29}$$

We remark that $\mathbf{g}_{1,\epsilon}$ and $\mathbf{g}_{2,\epsilon}$ determine the same plane as $\mathbf{g}^{1,\epsilon}$ and $\mathbf{g}^{2,\epsilon}$ (the normal vector of this plane is $\mathbf{a}_3$). Therefore, the boundedness of sequence $\{\mathbf{u}_\epsilon\cdot\mathbf{g}_{\alpha,\epsilon}\}_{\epsilon\in(0,1)}$ in $L^p\left(0,T;W^{1,p}(\Omega)\right)\cap L^2\left(0,T;L^2(\partial\Omega)\right)$ implies the boundedness of $\{\mathbf{u}_\epsilon\cdot\mathbf{g}^{\alpha,\epsilon}\}_{\epsilon\in(0,1)}$ in the same space, for $\alpha=1,2$.

There are three types of terms in (6.3.29) and we analyze "the worst one", i.e. term $\rho_\epsilon(\mathbf{u}_\epsilon\cdot\mathbf{a}_3)^2[\bar\omega_\epsilon(\psi)]_{33}$ (because $\mathbf{r}^{3,\epsilon}=\mathbf{a}_3$ see (4.5.9)). For convenience, let us denote $\epsilon^{-1}\varphi(t)\bar\psi(x) := [\bar\omega_\epsilon(\psi(x,t))]_{33}$, where $\varphi\in L^q(0,T)$, $2/p+1/q=1$, and $\bar\psi\in E_{\Psi_{\gamma-2}}(\Omega)^9$. By the use of Hölder's inequality, we get

$$\left|\int_0^T\int_\Omega \rho_\epsilon(\mathbf{u}_\epsilon\cdot\mathbf{a}_3)^2\epsilon^{-1}\varphi\bar\psi\sqrt{d_\epsilon}\;dxdt\right| \le$$
$$\le \|\sqrt{d_\epsilon}\|_\infty\|\epsilon^{-1}(\mathbf{u}_\epsilon\cdot\mathbf{a}_3)^2\|_{L_{\Psi_2}(\Omega\times(0,T))}\|\rho_\epsilon\varphi\bar\psi\|_{L_{\Phi_2}(\Omega\times(0,T))}.$$

The norm $\|\epsilon^{-1}(\mathbf{u}_\epsilon \cdot \mathbf{a}_3)^2\|_{L_{\Psi_2}(\Omega\times(0,T))}$ is bounded due to (6.3.12), because M and Ψ_1 are the equivalent Young functions, and $\Psi_2(z^2) \sim \Psi_1(z)$. We estimate the remaining norm $\|\rho_\epsilon\varphi\bar{\psi}\|_{L_{\Phi_2}(\Omega\times(0,T))}$ in the following way:

$$\begin{aligned}&\|\rho_\epsilon\varphi\bar{\psi}\|_{L_{\Phi_2}(\Omega\times(0,T))} \le \int_0^T\int_\Omega \Phi_2(\rho_\epsilon|\varphi||\bar{\psi}|)\ dxdt + C_1 \le \\ &\le \int_0^T\int_\Omega |\varphi||\bar{\psi}|\Phi_2(\rho_\epsilon)\ dxdt + \int_0^T\int_\Omega \rho_\epsilon\Phi_2(|\varphi||\bar{\psi}|)\ dxdt + \\ &+2\int_0^T\int_\Omega \Phi_1(\rho_\epsilon)\Phi_1(|\varphi||\bar{\psi}|)\ dxdt + C_2,\end{aligned}$$

where "the worst term" can be estimated as follows

$$\begin{aligned}&\int_0^T\int_\Omega |\varphi||\bar{\psi}|\Phi_2(\rho_\epsilon)\ dxdt \le \\ &\le C\|\varphi\|_{L^1(0,T)}\left(\left\|\int_\Omega \Phi_\gamma(\rho_\epsilon)\ dx\right\|_{L^\infty(0,T)} + \int_\Omega \Psi_{\gamma-2}(|\bar{\psi}|)\ dx + C\right).\end{aligned}$$

We conclude that the integral (6.3.28) is bounded for any $\epsilon \in (0,1)$ and $\psi \in L^q(0,T;W^1E_{\Psi_{\gamma-2}}(\Omega)^3)$.

Subsequently, we show that also

$$\int_0^T\int_\Omega P\left(|\omega_\epsilon(\mathbf{u}_\epsilon)|\right)\omega_\epsilon(\mathbf{u}_\epsilon) : \omega_\epsilon(\psi)\sqrt{d_\epsilon}\ dxdt \tag{6.3.30}$$

is bounded for any $\epsilon \in (0,1)$ and $\psi(x,t) = \varphi(t)\bar{\psi}(x)$, where $\varphi \in E_{\Psi_{1/\alpha}}(0,T)$, $\alpha > 2$, and $\bar{\psi} \in W^1E_{\Psi_{1/2}}(\Omega)^3$, $\partial_3\bar{\psi} = 0$. We remark that according to (6.2.1) and (6.2.2) we have

$$\omega_\epsilon(\bar{\psi}) = R_\epsilon^T\begin{pmatrix} \partial_1\bar{\psi}\cdot\mathbf{g}_{1,\epsilon} & \frac{1}{2}\left(\partial_1\bar{\psi}\cdot\mathbf{g}_{2,\epsilon} + \partial_2\bar{\psi}\cdot\mathbf{g}_{1,\epsilon}\right) & \frac{1}{2}\left(\partial_1\bar{\psi}\cdot\mathbf{a}_3\right) \\ \cdot & \partial_2\bar{\psi}\cdot\mathbf{g}_{2,\epsilon} & \frac{1}{2}\left(\partial_2\bar{\psi}\cdot\mathbf{a}_3\right) \\ \text{sym} & \cdot & 0\end{pmatrix}R_\epsilon,$$

which is bounded for $\epsilon \to 0$ in $E_{\Psi_{1/2}}(\Omega)^9$ due to (4.2.6), (4.2.9), Propostion 4.3.2, and Corollary 4.3.3. Due to Young's inequality, it holds that

$$\begin{aligned}&\left|\int_0^T\int_\Omega P\left(|\omega_\epsilon(\mathbf{u}_\epsilon)|\right)\omega_\epsilon(\mathbf{u}_\epsilon) : \omega_\epsilon(\bar{\psi})\varphi\sqrt{d_\epsilon}\ dxdt\right| \le \\ &\le \|\sqrt{d_\epsilon}\|_\infty\left(|\Omega|\int_0^T \Psi_{1/\alpha}(|\varphi|)\ dt + \right. \\ &\left. + \int_0^T\int_\Omega \Phi_{1/\alpha}\left(P(|\omega_\epsilon(\mathbf{u}_\epsilon)|)|\omega_\epsilon(\mathbf{u}_\epsilon)||\omega_\epsilon(\bar{\psi})|\right)\ dxdt\right),\end{aligned} \tag{6.3.31}$$

where $\alpha > 2$. For brevity, let us denote $w_\epsilon := P(|\omega_\epsilon(\mathbf{u}_\epsilon)|)|\omega_\epsilon(\mathbf{u}_\epsilon)|$. Since $w_\epsilon \in L_{\Phi_1}(\Omega\times(0,T))$ implies $w_\epsilon \in L_{\Phi_{(\alpha-1)/\alpha}}(0,T;L_{\Phi_{1/\alpha}}(\Omega))$, which follows from Jensen's inequality and estimate

$$\Phi_{(\alpha-1)/\alpha}(\Phi_{1/\alpha}(z)) \le 2\Phi_1(z) + C, \quad z \ge 0,$$

the second term on the right-hand side of (6.3.31) is less or equal than

$$\int_0^T \int_\Omega |\omega_\epsilon(\bar{\psi})|\Phi_{1/\alpha}(w_\epsilon) + w_\epsilon \Phi_{1/\alpha}(|\omega_\epsilon(\bar{\psi})|)\ dxdt \leq$$
$$\leq \int_0^T \int_\Omega \Phi_{(\alpha-1)/\alpha}(\Phi_{1/\alpha}(w_\epsilon)) + \Psi_{(\alpha-1)/\alpha}(|\omega_\epsilon(\bar{\psi})|) +$$
$$+\Phi_1(w_\epsilon) + \Psi_1(\Phi_{1/\alpha}(|\omega_\epsilon(\bar{\psi})|))\ dxdt \leq$$
$$\leq 3\int_0^T \int_\Omega \Phi_1\left(P(|\omega_\epsilon(\mathbf{u}_\epsilon)|)|\omega_\epsilon(\mathbf{u}_\epsilon)|\right)\ dxdt +$$
$$+\int_0^T \int_\Omega \Psi_{(\alpha-1)/\alpha}\left(|\omega_\epsilon(\bar{\psi})|\right)\ dxdt + \int_0^T \int_\Omega \Psi_{1/2}\left(|\omega_\epsilon(\bar{\psi})|\right)\ dxdt + C,$$

where $\alpha > 2$. Due to property (6.1.16), we conclude that integral (6.3.30) is bounded.

Terms (6.3.28) and (6.3.30) represent "the worst integrals" in (6.2.6). Thus, we omit the estimates of the others and take $\psi \in E_{\Psi_{\gamma_{1/\alpha}}}(0,T;W^1E_{\Psi_{\gamma_{1/2}}}(\Omega)^3)$ as a test function. By the use of estimates (6.3.28) and (6.3.30), we demonstrate how to perform a limit passage in the second term on the left-hand side of equation (6.2.6). Let us test the equation (6.2.6) by function $\psi(x,t) = \bar{\psi}(x)\varphi(t)$, where $\varphi \in C_0^\infty(0,T)$ and $\bar{\psi} \in W^1E_{\Psi_{1/2}}(\Omega)^3$, $\partial_3\bar{\psi} = 0$, $\bar{\psi}\cdot\mathbf{a}^2 = 0$, $\bar{\psi}\cdot\mathbf{a}_3 = 0$ (to control term $\nabla\bar{\psi} : R_\epsilon E_\epsilon$) and $\bar{\psi}\cdot\mathbf{n}|_{\partial\Omega} = 0$. Since $\mathbf{a}^3 = \mathbf{a}_3$ we have

$$\bar{\psi} = (\bar{\psi}\cdot\mathbf{a}^1)\mathbf{a}_1 + (\bar{\psi}\cdot\mathbf{a}^2)\mathbf{a}_2 + (\bar{\psi}\cdot\mathbf{a}^3)\mathbf{a}_3 = (\bar{\psi}\cdot\mathbf{a}^1)\mathbf{a}_1.$$

We thus get

$$\left|\int_0^T \varphi' \int_\Omega \rho_\epsilon((\mathbf{u}_\epsilon\cdot\mathbf{g}_{1,\epsilon})\mathbf{g}^{1,\epsilon} + (\mathbf{u}_\epsilon\cdot\mathbf{g}_{2,\epsilon})\mathbf{g}^{2,\epsilon})\cdot\bar{\psi}\ \sqrt{d_\epsilon}\ dxdt\right| \leq$$
$$\leq \int_0^T |\varphi| \int_\Omega \left[|\rho_\epsilon\mathbf{u}_\epsilon\otimes\mathbf{u}_\epsilon : \omega_\epsilon(\bar{\psi})| + |\rho_\epsilon\nabla\bar{\psi} : R_\epsilon E_\epsilon| + \right.$$
$$\left.+|P\left(|\omega_\epsilon(\mathbf{u}_\epsilon)|\right)\omega_\epsilon(\mathbf{u}_\epsilon) : \omega_\epsilon(\bar{\psi})| + |\rho_\epsilon\mathbf{f}_\epsilon\cdot\bar{\psi}|\right]\sqrt{d_\epsilon}\ dxdt +$$
$$+q\int_0^T |\varphi| \int_{\Gamma_1} |\mathbf{u}_\epsilon\cdot\bar{\psi}||R_\epsilon E_\epsilon\mathbf{n}|\sqrt{d_\epsilon}\ d\Gamma dt +$$
$$+\frac{h(\epsilon)}{\epsilon}\int_0^T |\varphi| \int_{\Gamma_2} |\mathbf{u}_\epsilon\cdot\bar{\psi}|\sqrt{d_\epsilon}\ d\Gamma dt, \tag{6.3.32}$$

where, due to (4.5.2) and (4.5.6),

$$\rho_\epsilon\left((\mathbf{u}_\epsilon\cdot\mathbf{g}_{1,\epsilon})\mathbf{g}^{1,\epsilon} + (\mathbf{u}_\epsilon\cdot\mathbf{g}_{2,\epsilon})\mathbf{g}^{2,\epsilon}\right)\cdot\bar{\psi} = \left(\rho_\epsilon(\mathbf{u}_\epsilon\cdot\mathbf{g}_{1,\epsilon})\mathbf{g}^{1,\epsilon} - \epsilon\mathbf{z}_{1,\epsilon}\right)\cdot\bar{\psi},$$

where $\mathbf{z}_{1,\epsilon} := \rho_\epsilon(\mathbf{u}_\epsilon\cdot\mathbf{g}_{2,\epsilon})(x_3\mathbf{g}^{2,\epsilon}\cdot\partial_1\mathbf{a}_3)\mathbf{a}^1$. The boundedness of $\{\mathbf{z}_{1,\epsilon}\}_{\epsilon\in(0,1)}$ in $L^p(0,T;L_{\Phi_\gamma}(\Omega)^3)$ follows from convergences (4.5.11), (4.5.15), (4.5.16), boundedness (6.3.9), and (6.3.15). Therefore, $\epsilon\mathbf{z}_{1,\epsilon}\to 0$ in $L^p(0,T;L_{\Phi_\gamma}(\Omega)^3)$, and thus also in $L_{\Phi_\gamma}(\Omega\times(0,T))^3$.

Considering the density of $C_0^\infty(0,T)$ in $E_{\Psi_{1/2}}(0,T)$, imbedding $L_{\Psi_{1/\alpha}}(0,T)\hookrightarrow E_{\Psi_{1/2}}(0,T)\subset\tilde{L}_{\Psi_{1/2}}(0,T)$, $\alpha > 2$, and the boundedness of all terms on the right-hand side of the inequality (6.3.32) (see (6.3.28) and (6.3.30)), we deduce the boundedness of

$$\left\{\partial_t\int_0^1 \left(\rho_\epsilon(\mathbf{u}_\epsilon\cdot\mathbf{g}_{1,\epsilon})\mathbf{g}^{1,\epsilon} - \epsilon\mathbf{z}_{1,\epsilon}\right)\sqrt{d_\epsilon}\ dx_3\right\}_{\epsilon\in(0,1)}$$
$$\text{in } L_{\Phi_{1/\alpha}}(0,T;[WE_{\Psi_{1/2}}(S)^3]'),\ \alpha > 2. \tag{6.3.33}$$

Similarly, testing the equation (6.2.6) by function $\psi(x,t) = \bar{\psi}(x)\varphi(t)$, where $\varphi \in C_0^\infty(0,T)$ and $\bar{\psi} \in W^1E_{\Psi_{1/2}}(\Omega)^3$, $\partial_3\bar{\psi} = 0$, $\bar{\psi}\cdot\mathbf{a}^1 = 0$, $\bar{\psi}\cdot\mathbf{a}_3 = 0$ and $\bar{\psi}\cdot\mathbf{n}|_{\partial\Omega} = 0$, leads to the boundedness of

$$\left\{\partial_t \int_0^1 \left(\rho_\epsilon(\mathbf{u}_\epsilon\cdot\mathbf{g}_{2,\epsilon})\mathbf{g}^{2,\epsilon} - \epsilon\mathbf{z}_{2,\epsilon}\right)\sqrt{d_\epsilon}\, dx_3\right\}_{\epsilon\in(0,1)}$$
$$\text{in } L_{\Phi_{1/\alpha}}(0,T;[WE_{\Psi_{1/2}}(S)^3]'),\ \alpha > 2, \tag{6.3.34}$$

where $\mathbf{z}_{2,\epsilon} = \rho_\epsilon(\mathbf{u}_\epsilon\cdot\mathbf{g}_{1,\epsilon})(x_3\mathbf{g}^{1,\epsilon}\cdot\partial_2\mathbf{a}_3)\mathbf{a}^2$.

By the use of (4.5.11), (4.5.15), (4.5.16), (6.3.25), (6.3.33), (6.3.34), and theorem concerning compact imbedding (see [188], Lemma 9), we get (passing to subsequences if necessary)

$$\int_0^1 \left(\rho_\epsilon(\mathbf{u}_\epsilon\cdot\mathbf{g}_{\alpha,\epsilon})\mathbf{g}^{\alpha,\epsilon} - \epsilon\mathbf{z}_{\alpha,\epsilon}\right)\sqrt{d_\epsilon}\, dx_3 \to \int_0^1 \rho(\mathbf{u}\cdot\mathbf{a}_\alpha)\mathbf{a}^\alpha\sqrt{d}\, dx_3$$
$$\text{in } C\left([0,T];[WL_{\Psi_1}(S)^3]'\right),\ \alpha = 1,2. \tag{6.3.35}$$

To perform a limit passage in the second term on the left-hand side of equation (6.2.6), we prove the following lemma.

Lemma 6.3.2 *Let us remind notation* $\mathbf{v}_\epsilon = (\mathbf{u}_\epsilon\cdot\mathbf{g}_{1,\epsilon}, \mathbf{u}_\epsilon\cdot\mathbf{g}_{2,\epsilon}, \mathbf{u}_\epsilon\cdot\epsilon\mathbf{a}_3)$. *Assume that* $\{\mathbf{u}_\epsilon\}_{\epsilon\in(0,1)}$ *satisfies condition* (6.3.8) *and* $\{\mathbf{v}_\epsilon\}_{\epsilon\in(0,1)}$ *satisfies condition* (6.3.9). *Then for any* $p > 3$ (*passing to a subsequence if necessary*), *it holds that*

$$\left\|v_{\alpha,\epsilon} - \int_0^1 v_{\alpha,\epsilon}\, dx_3\right\|_{L^p(0,T;L^\infty(\Omega))} \to 0,\ \textit{for } \epsilon\to 0 \textit{ and } \alpha = 1,2. \tag{6.3.36}$$

Proof: We prove the assertion by a contradiction in several steps. Let us suppose the existence of fixed $p > 3$ with a positive constant C_1 and $\{\epsilon_n\}_{n=1}^{+\infty}$ tending to zero such that

$$\left\|v_{\alpha,\epsilon_n} - \int_0^1 v_{\alpha,\epsilon_n}\, dx_3\right\|_{L^p(0,T;L^\infty(\Omega))} \geq C_1 > 0,\ \forall n\in\mathbb{N}. \tag{6.3.37}$$

Obviously, there exists a nonempty set $I_{\epsilon_n,C_1} \subset (0,T)$ such that

$$\left\|v_{\alpha,\epsilon_n}(t) - \int_0^1 v_{\alpha,\epsilon_n}(t)\, dx_3\right\|_\infty \geq \frac{C_1}{T^{1/p}},\ \forall t\in I_{\epsilon_n,C_1}. \tag{6.3.38}$$

(i) *There exists a positive constant* $C_2 = C_2(C_1)$ *such that* $|I_{\epsilon_n,C_1}| \geq C_2 > 0$, *for all* $n\in\mathbb{N}$.

If not, then (passing to a subsequence if necessary) $|I_{\epsilon_n,C_1}| \to 0$ for ϵ_n tending to zero. Let us consider $q\in\mathbb{R}$ such that $q > p$. Due to the boundedness of $\{\mathbf{v}_{\epsilon_n}\}_{n=1}^{+\infty}$ in $L^q(0,T;W^{1,p}(\Omega)^3)$ for any $q > p$ (see (6.3.9)), the following inequality contradicts the relation (6.3.37):

$$\left\|v_{\alpha,\epsilon_n} - \int_0^1 v_{\alpha,\epsilon_n}\, dx_3\right\|_{L^p(0,T;L^\infty(\Omega))} =$$
$$= \sqrt[p]{\int_{(0,T)\setminus I_{\epsilon_n,C_1}} \left\|v_{\alpha,\epsilon_n}(t) - \int_0^1 v_{\alpha,\epsilon_n}(t)\, dx_3\right\|_\infty^p dt + \int_{I_{\epsilon_n,C_1}} \|\cdot\|_\infty^p\, dt} <$$
$$< C_1 + \left\|v_{\alpha,\epsilon_n} - \int_0^1 v_{\alpha,\epsilon_n}\, dx_3\right\|_{L^q(0,T;L^\infty(\Omega))}^p |I_{\epsilon_n,C_1}|^{\frac{q-p}{q}} \overset{n\to+\infty}{\longrightarrow} C_1.$$

(ii) *We show that there exists a nonempty set $J_{\epsilon_n,C_3} \subset (0,T)$, where $C_3 > 0$, such that,*

$$\|v_{\alpha,\epsilon_n}(t)\|_{2,\partial\Omega} + \|D_{12}\mathbf{v}_{\epsilon_n}(t)\|_p \leq C_3, \text{ for almost all } t \in J_{\epsilon_n,C_3}, \tag{6.3.39}$$

where $D_{12}\mathbf{v}_{\epsilon_n}$ is 2×2 submatrix of $D\mathbf{v}_{\epsilon_n}$ constituted of the first two rows and columns.

If not, then without loss of generality there exists a sequence $\{C_3(n)\}_{n=1}^{+\infty}$, $C_3(n) \to +\infty$, such that

$$\|v_{\alpha,\epsilon_n}(t)\|_{2,\partial\Omega} + \|D_{12}\mathbf{v}_{\epsilon_n}(t)\|_p > C_3(n), \ \forall t \in (0,T),$$

which would be a contradiction with the boundedness of $\{\mathbf{v}_{\epsilon_n}\}_{n=1}^{+\infty}$.

(iii) *It holds that* $\sup_{n\in\mathbb{N}} |(0,T) \setminus J_{\epsilon_n,C_3}| \to 0$ *for* $C_3 \to +\infty$.

If not, then there exist a sequence $\{C_3(m)\}_{m=1}^{+\infty}$, $C_3(m) \to +\infty$, and a positive constant C_4 such that $\sup_{n\in\mathbb{N}} |(0,T) \setminus J_{\epsilon_n,C_3(m)}| \geq C_4 > 0$, $\forall C_3(m) \geq C_3(m_0)$, $m_0 \in \mathbb{N}$. It implies (passing to a subsequence of $\{\epsilon_n\}_{n=1}^{+\infty}$ if necessary)

$$\|v_{\alpha,\epsilon_n}(t)\|_{2,\partial\Omega} + \|D_{12}\mathbf{v}_{\epsilon_n}(t)\|_p > C_3(m), \ \forall t \in (0,T) \setminus J_{\epsilon_n,C_3(m)}, \ \forall n \geq n_0,$$

where $n_0 \in \mathbb{N}$, and we would get a contradiction with the boundedness of sequence $\{\mathbf{v}_{\epsilon_n}\}_{n=1}^{+\infty}$.

(iv) *For convenience, we simplify the notation $v_{\alpha,\epsilon_n} = v_{\alpha,\epsilon_n}(t_n) \in W^{1,p}(\Omega)$, where $t_n \in I_{\epsilon_n,C_1}$. We justify that*

$$\|D_3\mathbf{v}_{\epsilon_n}\|_p + \|v_{3,\epsilon_n}\|_{2,\partial\Omega} \to 0, \tag{6.3.40}$$

where

$$D_3\mathbf{v}_{\epsilon_n} = \begin{pmatrix} 0 & 0 & \frac{1}{2}(\partial_1 v_{3,\epsilon_n} + \partial_3 v_{1,\epsilon_n}) \\ \cdot & 0 & \frac{1}{2}(\partial_2 v_{3,\epsilon_n} + \partial_3 v_{2,\epsilon_n}) \\ sym & \cdot & \partial_3 v_{3,\epsilon_n} \end{pmatrix}.$$

Comparing $D_3\mathbf{v}_{\epsilon_n}$ and (6.2.2), the statement of this step follows from definitions of $\mathbf{v}_{\epsilon_n}$ and $D_3\mathbf{v}_{\epsilon_n}$, and boundedness (6.3.8), (6.3.9), (6.3.12) for almost all $t_n \in I_{\epsilon_n,C_1}$.

(v) *According to parts (ii) and (iii), $|I_{\epsilon_n,C_1} \setminus J_{\epsilon_n,C_3}|$ tends to zero for $C_3 \to +\infty$. Therefore, $|I_{\epsilon_n,C_1} \cap J_{\epsilon_n,C_3}| \to |I_{\epsilon_n,C_1}|$ for $C_3 \to +\infty$. Hence, we can assume that both conditions* (6.3.38) *and* (6.3.39) *hold for almost all $t \in I_{\epsilon_n,C_1}$.*

We prove that

$$\left\| v_{\alpha,\epsilon_n} - \int_0^1 v_{\alpha,\epsilon_n} \, dx_3 \right\|_\infty \leq C(\|D_3\mathbf{v}_{\epsilon_n}\|_p + \|v_{3,\epsilon_n}\|_{2,\partial\Omega}), \tag{6.3.41}$$

where $C = C(C_1, C_3) > 0$ and using the notation from (iv).

There are two options for the behavior of $\left\| v_{\alpha,\epsilon_n} - \int_0^1 v_{\alpha,\epsilon_n} \, dx_3 \right\|_\infty$. First, let us assume that

$$\left\| v_{\alpha,\epsilon_n} - \int_0^1 v_{\alpha,\epsilon_n} \, dx_3 \right\|_\infty \to +\infty, \text{ for } n \to +\infty.$$

For contradiction with (6.3.41), we further suppose that

$$C_{\epsilon_n} := \left\| v_{\alpha,\epsilon_n} - \int_0^1 v_{\alpha,\epsilon_n} \, dx_3 \right\|_\infty > n(\|D_3\mathbf{v}_{\epsilon_n}\|_p + \|v_{3,\epsilon_n}\|_{2,\partial\Omega}).$$

Dividing this inequality by C_{ϵ_n} leads to

$$1 = \left\| w_{\alpha,\epsilon_n} - \int_0^1 w_{\alpha,\epsilon_n}\ dx_3 \right\|_\infty > n(\|D_3\mathbf{w}_{\epsilon_n}\|_p + \|w_{3,\epsilon_n}\|_{2,\partial\Omega}),$$

where $\mathbf{w}_{\epsilon_n} = C_{\epsilon_n}^{-1}\mathbf{v}_{\epsilon_n}$. We divide also (6.3.39) by C_{ϵ_n} and together with (6.3.40) we get the convergences $D\mathbf{w}_{\epsilon_n} \to 0$ in $L^p(\Omega)^9$ and $w_{\alpha,\epsilon_n} \to 0$ in $L^2(\partial\Omega)$. From Korn's inequality (see (6.3.2)), we conclude that $w_{\alpha,\epsilon_n} \to 0$ in $W^{1,p}(\Omega)$ (and also in $L^\infty(\Omega)$ from the compact imbedding), which is a contradiction with the unit norm of $w_{\alpha,\epsilon_n} - \int_0^1 w_{\alpha,\epsilon_n}\ dx_3$.

Second, let us suppose that

$$\left\| v_{\alpha,\epsilon_n} - \int_0^1 v_{\alpha,\epsilon_n}\ dx_3 \right\|_\infty \leq C_5 < +\infty,\ \forall n \in \mathbb{N}.$$

For contradiction with (6.3.41), we further assume that

$$C_5 \geq \left\| v_{\alpha,\epsilon_n} - \int_0^1 v_{\alpha,\epsilon_n}\ dx_3 \right\|_\infty > n(\|D_3\mathbf{v}_{\epsilon_n}\|_p + \|v_{3,\epsilon_n}\|_{2,\partial\Omega}). \tag{6.3.42}$$

Considering inequalities (6.3.39), (6.3.42) and classical Korn's inequality, we arrive at the boundedness of $\{\|v_{\alpha,\epsilon_n}\|_\infty\}_{n=1}^{+\infty}$. Therefore (passing to a subsequence if necessary), it follows from the compact imbedding that $v_{\alpha,\epsilon_n} \to v_\alpha$ in $L^\infty(\Omega)$.

Due to (6.3.40), $\partial_3 v_{3,\epsilon_n} \to 0$ in $L^p(\Omega)$, which, together with the convergence $v_{3,\epsilon_n} \to 0$ in $L^2(\partial\Omega)$, gives us $v_{3,\epsilon_n} \to 0$ in $L^p(\Omega)$ (we remind that $\Omega = S \times (0,1)$). Hence, $\partial_\alpha v_{3,\epsilon} \to 0$ in $\mathcal{D}'(\Omega)$ and also $\partial_3 v_{\alpha,\epsilon} \to 0$ in $\mathcal{D}'(\Omega)$ due to the convergence (6.3.40). To conclude, $\partial_3 v_{\alpha,\epsilon} \to \partial_3 v_\alpha = 0$ implies $v_\alpha = \int_0^1 v_\alpha\ dx_3$, which contradicts the inequality (6.3.38).

(vi) *Since convergence* (6.3.40) *and inequality* (6.3.41) *hold* (*see steps* (*iv*) *and* (*v*)), *we arrive at a contradiction with inequality* (6.3.38). *It means that the statement of this lemma holds true.*

□

We apply Lemma 6.3.2 in the following way. Let us remark that $\mathbf{v}_\epsilon = (\mathbf{u}_\epsilon \cdot \mathbf{g}_{1,\epsilon}, \mathbf{u}_\epsilon \cdot \mathbf{g}_{2,\epsilon}, \mathbf{u}_\epsilon \cdot \epsilon\mathbf{a}_3)$. Then,

$$\begin{aligned}
&\int_0^T \int_S \int_0^1 \rho_\epsilon v_{\alpha,\epsilon} v_{\beta,\epsilon} g^{\alpha\beta,\epsilon} \psi \sqrt{d_\epsilon}\ dx_3 d\hat{x} dt = \\
&= \int_0^T \int_S \int_0^1 \rho_\epsilon v_{\alpha,\epsilon}(v_{\beta,\epsilon} - \int_0^1 v_{\beta,\epsilon}\ dx_3) g^{\alpha\beta,\epsilon} \psi \sqrt{d_\epsilon}\ dx_3 d\hat{x} dt + \\
&+ \int_0^T \int_S \left(\int_0^1 \rho_\epsilon v_{\alpha,\epsilon} g^{\alpha\beta,\epsilon} \sqrt{d_\epsilon}\ dx_3 \right) \left(\int_0^1 v_{\beta,\epsilon}\ dx_3 \right) \psi\ d\hat{x} dt,
\end{aligned}$$

where $\psi \in C_0^\infty(0,T; C^\infty(\bar{\Omega}))$, $\partial_3\psi = 0$, $d\hat{x} = dx_1 dx_2$, and α, $\beta = 1, 2$. The first integral on the right-hand side tends to zero for $\epsilon \to 0$ due to Lemma 6.3.2. Concerning the second integral on the right-hand side, it holds that (due to convergences (6.3.35))

$$\begin{aligned}
&\int_0^1 \rho_\epsilon v_{\alpha,\epsilon} g^{\alpha\beta,\epsilon} \sqrt{d_\epsilon}\ dx_3 = \int_0^1 (\rho_\epsilon v_{\alpha,\epsilon} g^{\alpha\beta,\epsilon} - \epsilon \mathbf{z}_{\alpha,\epsilon} \cdot \mathbf{g}^{\beta,\epsilon}) \sqrt{d_\epsilon}\ dx_3 + \\
&+\epsilon \int_0^1 \mathbf{z}_{\alpha,\epsilon} \cdot \mathbf{g}^{\beta,\epsilon} \sqrt{d_\epsilon}\ dx_3 \to \int_0^1 \rho_\epsilon v_\alpha g^{\alpha\beta} \sqrt{d}\ dx_3,
\end{aligned} \tag{6.3.43}$$

where the first integral converges in $C([0,T];[WL_{\Psi_1}(S)]')$ and the second integral converges in $L^p(0,T;L_{\Phi_\gamma}(S))$, and also

$$\int_0^1 v_{\beta,\epsilon}\ dx_3 \rightharpoonup \int_0^1 v_\beta \ \ dx_3 \quad \text{in } L^p(0,T;W^{1,p}(S)),$$

which follows from (6.3.21). In addition, it holds that

$$\begin{aligned}\int_0^1 \rho v_\alpha g^{\alpha\beta}\sqrt{d}\ dx_3 &= \hat{\rho} v_\alpha g^{\alpha\beta}\sqrt{d}\\ \int_0^1 v_\beta\ dx_3 &= v_\beta,\end{aligned}$$

where $\hat{\rho} := \int_0^1 \rho\ dx_3$, because $\mathbf{v}$ (as well as $\mathbf{u}$) is independent of x_3. Hence, convergences

$$\int_0^T\int_\Omega \rho_\epsilon(\mathbf{u}_\epsilon\cdot\mathbf{g}_{\alpha,\epsilon})(\mathbf{u}_\epsilon\cdot\mathbf{g}_{\beta,\epsilon})g^{\alpha\beta,\epsilon}\psi\sqrt{d_\epsilon}\ dxdt \to$$
$$\to \int_0^T\int_S \hat{\rho}(\mathbf{u}\cdot\mathbf{a}_\alpha)(\mathbf{u}\cdot\mathbf{a}_\beta)g^{\alpha\beta}\psi\sqrt{d}\ d\hat{x}dt,\ \alpha,\beta=1,2, \tag{6.3.44}$$

are immediate consequences of (6.3.21), (6.3.35), and (6.3.36). Convergences (6.3.44) are applied in the next section to overcome the nonlinearity in the second term on the left-hand side of (6.2.6).

6.3.3 Limit of the governing equations

We prescribe the behavior of initial states for $\epsilon\to 0$ by formulas

$$\int_0^1 \rho_{0,\epsilon}\ln(\rho_{0,\epsilon})\sqrt{d_\epsilon}\ dx_3 \ \to\ \rho_0\ln(\rho_0)\sqrt{d} \quad \text{in } L^1(S), \tag{6.3.45}$$

$$\int_0^1 \Phi_\gamma(\rho_{0,\epsilon})\sqrt{d_\epsilon}\ dx_3 \ \to\ \Phi_\gamma(\rho_0)\sqrt{d} \quad \text{in } L^1(S),\ \gamma>3, \tag{6.3.46}$$

$$\int_0^1 \frac{|(\rho_\epsilon\mathbf{u}_\epsilon)_0|^2}{2\rho_{0,\epsilon}}\sqrt{d_\epsilon}\ dx_3 \ \to\ \frac{|(\rho\mathbf{u})_0|^2}{2\rho_0}\sqrt{d} \quad \text{in } L^1(S), \tag{6.3.47}$$

where all limits on the right-hand sides do not depend on x_3. We remark that the prescribed behavior (6.3.46) enables us to use Gronwall's lemma in the proof of boundedness (6.3.15). Further, we assume that $h(\epsilon)>0$ in (6.2.6) satisfies the condition $h(\epsilon)\sim O(\epsilon)$ to assure the convergence of $\frac{h(\epsilon)}{\epsilon}$ to a real positive number.

In this section, we denote a mean value of a function in the third spatial variable over interval $(0,1)$ by symbol " $\hat{}$ " over the function. Obviously, these mean values depend only on x_1 and x_2. For example, we write $\hat{\rho} := \int_0^1 \rho\ dx_3$.

Now, we can perform the limit in (6.2.5) and (6.2.6). We use convergences (4.5.11)–(4.5.17). Let us denote $\hat{\mathbf{u}} = (\mathbf{u}\cdot\mathbf{a}_1)\mathbf{a}^1 + (\mathbf{u}\cdot\mathbf{a}_2)\mathbf{a}^2$. Since $\mathbf{u}$ is independent of x_3 (see (6.3.24)), $\hat{\mathbf{u}}$ depends only on x_1 and x_2 and thus we do not contradict the notation above. First, we test the equation (6.2.5) by function $\varphi\in\mathcal{D}(\mathbb{R}^2\times[0,T])$. We arrive at

$$\int_0^T\int_\Omega \left[\rho_\epsilon\partial_t\varphi + \rho_\epsilon\mathbf{u}_\epsilon^T(\mathbf{g}^{1,\epsilon},\mathbf{g}^{2,\epsilon},\mathbf{g}^{3,\epsilon})(\partial_1\varphi,\partial_2\varphi,0)^T\right]\sqrt{d_\epsilon}\ dxdt = 0.$$

Subsequently, we expand $\mathbf{u}_\epsilon$ into the covariant basis. Since $\mathbf{g}^{\alpha,\epsilon}\cdot\mathbf{a}_3 = 0$, for $\alpha=1,2$, we obtain

$$\int_0^T\int_\Omega \left[\rho_\epsilon\partial_t\varphi + \rho_\epsilon[(\mathbf{u}_\epsilon\cdot\mathbf{g}_{1,\epsilon})\mathbf{g}^{1,\epsilon} + (\mathbf{u}_\epsilon\cdot\mathbf{g}_{2,\epsilon})\mathbf{g}^{2,\epsilon}]^T(\mathbf{g}^{1,\epsilon},\mathbf{g}^{2,\epsilon})\hat{\nabla}\varphi\right]\sqrt{d_\epsilon}\ dxdt = 0,$$

where $\hat{\nabla}\varphi := (\partial_1\varphi, \partial_2\varphi)$. Afterwards, we perform the limit for $\epsilon \to 0$, apply convergence (6.3.25) and get

$$\int_0^T \int_S \left[\hat{\rho}\partial_t\varphi + \hat{\rho}\hat{\mathbf{u}}^T R^{12}\hat{\nabla}\varphi\right] \sqrt{d}\; d\hat{x}dt = 0, \tag{6.3.48}$$

for any $\varphi \in \mathcal{D}\left(\mathbb{R}^2 \times [0,T]\right)$, where $R^{12} := (\mathbf{a}^1, \mathbf{a}^2)$ is a submatrix of R and $d\hat{x} = dx_1 dx_2$.

Second, we test the equation (6.2.6) by function $\psi \in C_0^\infty(0,T; C^\infty(\bar{\Omega})^3)$ such that $\psi \cdot \mathbf{a}_3 = 0$, $\partial_3\psi = 0$ and $\psi \cdot \mathbf{n}|_{\partial S \times (0,T)} = 0$. We will show the limit passage for each term in (6.2.6) independently.

(a) $\rho_\epsilon \mathbf{u}_\epsilon \cdot \partial_t\psi$

We expand $\mathbf{u}_\epsilon$ into the covariant basis. Since $\psi \cdot \mathbf{a}_3 = 0$ and convergences (6.3.25) hold, we get

$$\int_0^T \int_\Omega \rho_\epsilon \mathbf{u}_\epsilon \cdot \partial_t\psi\sqrt{d_\epsilon}\; dx\; dt = \int_0^T \int_\Omega \rho_\epsilon [(\mathbf{u}_\epsilon \cdot \mathbf{g}_{1,\epsilon})\mathbf{g}^{1,\epsilon} + (\mathbf{u}_\epsilon \cdot \mathbf{g}_{2,\epsilon})\mathbf{g}^{2,\epsilon}] \cdot \partial_t\psi\sqrt{d_\epsilon}\; dxdt,$$

which converges to

$$\int_0^T \int_S \hat{\rho}\hat{\mathbf{u}} \cdot \partial_t\psi\sqrt{d}\; d\hat{x}dt,$$

for $\epsilon \to 0$, due to (6.3.25).

(b) $\rho_\epsilon \mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon : \omega_\epsilon(\psi)$

As $\partial_3\psi = 0$ and $\psi \cdot \mathbf{a}_3 = 0$, we know that $[\bar{\omega}_\epsilon(\psi)]_{33} = 0$ and also that $[\bar{\omega}_\epsilon(\psi)]_{\alpha 3} = (\partial_\alpha\psi \cdot \mathbf{a}_3)/2$, $\alpha = 1, 2$. After expanding $\mathbf{u}_\epsilon$ into the covariant basis and applying convergences (6.3.26) and (6.3.44), we conclude that

$$\begin{aligned}
&\int_0^T \int_\Omega \rho_\epsilon \mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon : \omega_\epsilon(\psi)\sqrt{d_\epsilon}\; dxdt = \\
&= \int_0^T \int_\Omega \rho_\epsilon (\mathbf{u}_\epsilon \cdot \mathbf{g}_{i,\epsilon})(\mathbf{u}_\epsilon \cdot \mathbf{g}_{j,\epsilon}) g^{ij,\epsilon} [\omega_\epsilon(\psi)]_{ij}\sqrt{d_\epsilon}\; dxdt,
\end{aligned}$$

where the sum is taken over i, $j = 1, 2, 3$, converges to

$$\begin{aligned}
&\int_0^T \int_S \hat{\rho}(\mathbf{u} \cdot \mathbf{a}_\alpha)(\mathbf{u} \cdot \mathbf{a}_\beta) g^{\alpha\beta} [\omega(\psi)]_{\alpha\beta}\sqrt{d}\; dxdt = \\
&= \int_0^T \int_S \hat{\rho}\hat{\mathbf{u}} \otimes \hat{\mathbf{u}} : \omega(\psi)\sqrt{d}\; d\hat{x}dt,
\end{aligned}$$

for $\epsilon \to 0$ (the sum is taken over α, $\beta = 1, 2$), where

$$\omega(\psi) = R^T \begin{pmatrix} \partial_1\psi \cdot \mathbf{a}_1 & \frac{1}{2}\left(\partial_1\psi \cdot \mathbf{a}_2 + \partial_2\psi \cdot \mathbf{a}_1\right) & \frac{1}{2}\partial_1\psi \cdot \mathbf{a}_3 \\ \cdot & \partial_2\psi \cdot \mathbf{a}_2 & \frac{1}{2}\partial_2\psi \cdot \mathbf{a}_3 \\ \text{sym} & \cdot & 0 \end{pmatrix} R.$$

(c) $\rho_\epsilon \nabla\psi : R_\epsilon E_\epsilon$

Since $R_\epsilon E_\epsilon = (\mathbf{g}^{1,\epsilon}, \mathbf{g}^{2,\epsilon}, \epsilon^{-1}\mathbf{a}_3)$, we have

$$\begin{aligned}
&\int_0^T \int_\Omega \rho_\epsilon (\partial_1\psi, \partial_2\psi, \partial_3\psi) : (\mathbf{g}^{1,\epsilon}, \mathbf{g}^{2,\epsilon}, \epsilon^{-1}\mathbf{a}_3)\sqrt{d_\epsilon}\; dxdt = \\
&= \int_0^T \int_\Omega \rho_\epsilon (\partial_1\psi, \partial_2\psi) : (\mathbf{g}^{1,\epsilon}, \mathbf{g}^{2,\epsilon})\sqrt{d_\epsilon}\; dxdt,
\end{aligned}$$

which tends to (see convergence (6.3.18))

$$\int_0^T \int_S \hat{\rho}\hat{\nabla}\psi : R^{12}\sqrt{d}\ d\hat{x}dt,$$

for $\epsilon \to 0$.

(d) $P\left(|\omega_\epsilon(\mathbf{u}_\epsilon)|\right)\omega_\epsilon(\mathbf{u}_\epsilon) : \omega_\epsilon(\psi)$

It holds that

$$\int_0^T \int_\Omega P\left(|\omega_\epsilon(\mathbf{u}_\epsilon)|\right)\omega_\epsilon(\mathbf{u}_\epsilon) : \omega_\epsilon(\psi)\sqrt{d_\epsilon}\ dxdt \to \int_0^T \int_\Omega \overline{P(|\zeta|)\zeta} : \omega(\psi)\sqrt{d}\ dxdt,$$

for $\epsilon \to 0$ due to (6.1.16) and (6.3.8), where ζ is defined by (6.3.22). Later, we will show that

$$\int_0^t \int_\Omega \overline{P(|\zeta|)\zeta} : \omega(\psi)\sqrt{d}\ dxds = \int_0^t \int_S P(|\omega(\hat{\mathbf{u}})|)\omega(\hat{\mathbf{u}}) : \omega(\psi)\sqrt{d}\ d\hat{x}ds,$$

for any $t \in (0, T)$.

(e) $\rho_\epsilon \mathbf{f}_\epsilon \cdot \psi$

After expanding $\mathbf{f}_\epsilon$ into the contravariant basis and applying the relation $\psi \cdot \mathbf{a}_3 = 0$, we arrive at

$$\int_0^T \int_\Omega \rho_\epsilon \mathbf{f}_\epsilon \cdot \psi\sqrt{d_\epsilon}\ dxdt = \int_0^T \int_\Omega \rho_\epsilon[(\mathbf{f}_\epsilon \cdot \mathbf{g}^{1,\epsilon})\mathbf{g}_{1,\epsilon} + (\mathbf{f}_\epsilon \cdot \mathbf{g}^{2,\epsilon})\mathbf{g}_{2,\epsilon}] \cdot \psi\sqrt{d_\epsilon}\ dxdt,$$

which tends to

$$\int_0^T \int_S \widehat{\rho\mathbf{F}} \cdot \psi\sqrt{d}\ d\hat{x}dt,$$

for $\epsilon \to 0$, where $\mathbf{F} = (\mathbf{f} \cdot \mathbf{a}^1)\mathbf{a}_1 + (\mathbf{f} \cdot \mathbf{a}^2)\mathbf{a}_2$ and $\mathbf{f}$ denotes the limit of $\mathbf{f}_\epsilon$.

(f) $\mathbf{u}_\epsilon \cdot \psi|R_\epsilon E_\epsilon \mathbf{n}|$

Since $\mathbf{n} = (n_1, n_2, 0)^T$ on Γ_1, we have

$$\mathbf{u}_\epsilon \cdot \psi|R_\epsilon E_\epsilon \mathbf{n}| = \mathbf{u}_\epsilon \cdot \psi|(\mathbf{g}^{1,\epsilon}, \mathbf{g}^{2,\epsilon})\hat{\mathbf{n}}|,$$

where $\hat{\mathbf{n}} = (n_1, n_2)$. Due to (6.3.21), we arrive at

$$\int_0^T \int_{\Gamma_1} \mathbf{u}_\epsilon \cdot \psi|R_\epsilon E_\epsilon \mathbf{n}|\sqrt{d_\epsilon}\ d\Gamma dt \to \int_0^T \int_{\partial S} \hat{\mathbf{u}} \cdot \psi|R^{12}\hat{\mathbf{n}}|\sqrt{d}\ dSdt,$$

as ϵ tends to zero.

(g) $\frac{h(\epsilon)}{\epsilon}\mathbf{u}_\epsilon \cdot \psi$

According to the supposed behavior of $h(\epsilon)$, i.e. $h(\epsilon) \sim O(\epsilon)$, we can use convergences (6.3.21) and get

$$\epsilon^{-1}\int_0^T \int_{\Gamma_2} h(\epsilon)\mathbf{u}_\epsilon \cdot \psi\sqrt{d_\epsilon}\ d\Gamma dt \to 2h\int_0^T \int_S \hat{\mathbf{u}} \cdot \psi\sqrt{d}\ d\hat{x}dt,$$

for $\epsilon \to 0$, where h is a positive constant.

Finally, we arrive at

$$\begin{aligned}
&\int_0^T \int_S \left[\hat{\rho}\hat{\mathbf{u}} \cdot \partial_t\psi + \hat{\rho}\hat{\mathbf{u}} \otimes \hat{\mathbf{u}} : \omega(\psi) + \hat{\rho}\hat{\nabla}\psi : R^{12}\right]\sqrt{d}\ d\hat{x}dt = \\
&= \int_0^T \int_\Omega \overline{P(|\zeta|)\zeta} : \omega(\psi)\sqrt{d}\ dxdt - \int_0^T \int_S \widehat{\rho\mathbf{F}} \cdot \psi\sqrt{d}\ d\hat{x}dt + \\
&+ q\int_0^T \int_{\partial S} \hat{\mathbf{u}} \cdot \psi|R^{12}\hat{\mathbf{n}}|\sqrt{d}\ dSdt + 2h\int_0^T \int_S \hat{\mathbf{u}} \cdot \psi\sqrt{d}\ d\hat{x}dt, \qquad (6.3.49)
\end{aligned}$$

for $\psi \in C_0^\infty(0, T; C^\infty(\bar{\Omega})^3)$ such that $\psi \cdot \mathbf{a}_3 = 0$, $\partial_3\psi = 0$ and $\psi \cdot \mathbf{n}|_{\partial S \times (0,T)} = 0$.

6.3.4 Limit of the energy equality

Applying similar approach as in Section 5.3.3, we perform the limit for $\epsilon \to 0$ also in the energy equality (6.2.9). We arrive at the following inequality:

$$\begin{aligned}
&\int_S \left[\hat{\rho}\frac{|\hat{\mathbf{u}}|^2}{2} + \hat{\rho}\ln(\hat{\rho})\right]\sqrt{d}\, d\hat{x} + \int_0^t \int_\Omega \overline{P(|\zeta|)\,|\zeta|^2}\sqrt{d}\, dxds + \\
&+q\int_0^t \int_{\partial S} |\hat{\mathbf{u}}|^2 |R^{12}\hat{\mathbf{n}}|\sqrt{d}\, dSds + 2h\int_0^t \int_S |\hat{\mathbf{u}}|^2\sqrt{d}\, d\hat{x}ds \leq \\
&\leq \int_0^t \int_S \widehat{\rho \mathbf{F}}\cdot \hat{\mathbf{u}}\sqrt{d}\, d\hat{x}ds + \int_S \frac{|(\rho\mathbf{u})_0|^2}{2\rho_0}\sqrt{d}\, d\hat{x} + \int_S \rho_0\ln(\rho_0)\sqrt{d}\, d\hat{x}.
\end{aligned} \tag{6.3.50}$$

By the use of the same procedure as in [127], Lemmas 3.2 and 3.3, based on the renormalized continuity equation and the Steklov function, we can derive from (6.3.48) and (6.3.49) the energy equality

$$\begin{aligned}
&\int_S \left[\hat{\rho}\frac{|\hat{\mathbf{u}}|^2}{2} + \hat{\rho}\ln(\hat{\rho})\right]\sqrt{d}\, d\hat{x} + \int_0^t \int_\Omega \overline{P(|\zeta|)\,\zeta} : \omega(\hat{\mathbf{u}})\sqrt{d}\, dxds + \\
&+q\int_0^t \int_{\partial S} |\hat{\mathbf{u}}|^2 |R^{12}\hat{\mathbf{n}}|\sqrt{d}\, dSds + 2h\int_0^t \int_S |\hat{\mathbf{u}}|^2\sqrt{d}\, d\hat{x}ds = \\
&= \int_0^t \int_S \widehat{\rho \mathbf{F}}\cdot \hat{\mathbf{u}}\sqrt{d}\, d\hat{x}ds + \int_S \frac{|(\rho\mathbf{u})_0|^2}{2\rho_0}\sqrt{d}\, d\hat{x} + \int_S \rho_0\ln(\rho_0)\sqrt{d}\, d\hat{x}.
\end{aligned} \tag{6.3.51}$$

Since the function $P(|z|)z$ is monotone (see (6.1.14)), we get

$$\begin{aligned}
0 \ &\leq \lim_{\epsilon\to 0}\int_0^t \int_\Omega (P(|\omega_\epsilon(\mathbf{u}_\epsilon)|)\omega_\epsilon(\mathbf{u}_\epsilon) - P(|T|)T) : (\omega_\epsilon(\mathbf{u}_\epsilon) - T)\ dxds = \\
&= \lim_{\epsilon\to 0}\int_0^t \int_\Omega P(|\omega_\epsilon(\mathbf{u}_\epsilon)|)|\omega_\epsilon(\mathbf{u}_\epsilon)|^2\ dxds - \\
&- \int_0^t \int_\Omega \overline{P(|\zeta|)\zeta} : T + P(|T|)T : \zeta + P(|T|)|T|^2\ \ dxds
\end{aligned} \tag{6.3.52}$$

for any symmetric $T \in \widetilde{L}_M(\Omega\times(0,T))^9$. As a consequence of (6.2.9), (6.3.51), convexity, and Jensen's inequality, we arrive at

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_0^t \int_\Omega P\left(|\omega_\epsilon(\mathbf{u}_\epsilon)|\right)|\omega_\epsilon(\mathbf{u}_\epsilon)|^2\ dxds = \\
&= \lim_{\epsilon\to 0}\left(-\int_\Omega \left[\rho_\epsilon\frac{|\mathbf{u}_\epsilon|^2}{2} + \rho_\epsilon\ln(\rho_\epsilon)\right]\sqrt{d_\epsilon}\ dx - \right.\\
&-q\int_0^t \int_{\Gamma_1} |\mathbf{u}_\epsilon|^2 |R_\epsilon E_\epsilon \mathbf{n}|\sqrt{d_\epsilon}\ d\Gamma ds - \frac{h(\epsilon)}{\epsilon}\int_0^t \int_{\Gamma_2} |\mathbf{u}_\epsilon|^2\sqrt{d_\epsilon}\ d\Gamma ds + \\
&+\int_0^t \int_\Omega \rho_\epsilon \mathbf{f}_\epsilon\cdot \mathbf{u}_\epsilon\sqrt{d_\epsilon}\ dxds + \int_\Omega \frac{|(\rho_\epsilon\mathbf{u}_\epsilon)_0|^2}{2\rho_{\epsilon,0}}\sqrt{d_\epsilon}\ dx + \\
&\left.+\int_\Omega \rho_{\epsilon,0}\ln(\rho_{\epsilon,0})\sqrt{d_\epsilon}\ dx\right) \leq -\int_S \left[\hat{\rho}\frac{|\hat{\mathbf{u}}|^2}{2} + \hat{\rho}\ln(\hat{\rho})\right]\sqrt{d}\ d\hat{x} - \\
&-q\int_0^t \int_{\partial S} |\hat{\mathbf{u}}|^2 |R^{12}\hat{\mathbf{n}}|\sqrt{d}\ dSds - 2h\int_0^t \int_S |\hat{\mathbf{u}}|^2\sqrt{d}\ d\hat{x}ds + \\
&+\int_0^t \int_S \widehat{\rho \mathbf{F}}\cdot \hat{\mathbf{u}}\sqrt{d}\ d\hat{x}ds + \int_S \frac{|(\rho\mathbf{u})_0|^2}{2\rho_0}\sqrt{d}\ d\hat{x} + \\
&+\int_S \rho_0\ln(\rho_0)\sqrt{d}\ d\hat{x} = \int_0^t \int_\Omega \overline{P(|\zeta|)\,\zeta} : \omega(\hat{\mathbf{u}})\sqrt{d}\ dxds.
\end{aligned} \tag{6.3.53}$$

Consequently from (6.3.52), we get

$$0 \le \int_0^t \int_\Omega \left(\overline{P(|\zeta|)\zeta} - P(|T|)T \right) : (\omega(\hat{\mathbf{u}}) - T) \ dxds.$$

Taking $T = \zeta + \lambda\omega(\psi)$ and $T = \zeta - \lambda\omega(\psi)$, for $\lambda > 0$, $\psi \in C_0^\infty(0,T; C^\infty(\bar{\Omega})^3)$ such that $\psi \cdot \mathbf{a}_3 = 0$, $\partial_3 \psi = 0$, and $\psi \cdot \mathbf{n}|_{\partial S \times (0,T)} = 0$, we conclude that

$$\int_0^t \int_\Omega \overline{P(|\zeta|)\zeta} : \omega(\psi) \ dxds = \int_0^t \int_S P(|\omega(\hat{\mathbf{u}})|)\omega(\hat{\mathbf{u}}) : \omega(\psi) \ d\hat{x}ds. \tag{6.3.54}$$

6.4 Conclusions

To sum it up, the limit equations together with the energy equality are given by the following formulas

$$\int_0^T \int_S \left[\hat{\rho}\partial_t \varphi + \hat{\rho}\hat{\mathbf{u}}^T R^{12} \hat{\nabla}\varphi \right] \sqrt{d} \ d\hat{x}dt = 0, \tag{6.4.1}$$

for any $\varphi \in \mathcal{D}\left(\mathbb{R}^2 \times [0,T]\right)$,

$$\begin{aligned} &\int_0^T \int_S \left[\hat{\rho}\hat{\mathbf{u}} \cdot \partial_t \psi + \hat{\rho}\hat{\mathbf{u}} \otimes \hat{\mathbf{u}} : \omega(\psi) + \hat{\rho}\hat{\nabla}\psi : R^{12} \right] \sqrt{d} \ d\hat{x}dt = \\ &= \int_0^T \int_S P(|\omega(\hat{\mathbf{u}})|)\omega(\hat{\mathbf{u}}) : \omega(\psi)\sqrt{d} \ d\hat{x}dt - \int_0^T \int_S \widehat{\rho \mathbf{F}} \cdot \psi \sqrt{d} \ d\hat{x}dt + \\ &+q \int_0^T \int_{\partial S} \hat{\mathbf{u}} \cdot \psi |R^{12}\hat{\mathbf{n}}| \sqrt{d} \ dSdt + 2h \int_0^T \int_S \hat{\mathbf{u}} \cdot \psi \sqrt{d} \ d\hat{x}dt, \end{aligned} \tag{6.4.2}$$

for any $\psi \in C_0^\infty(0,T; C^\infty(\Omega)^3)$ such that $\partial_3 \psi = 0$, $\psi \cdot \mathbf{a}_3 = 0$ in $\Omega \times (0,T)$ and $\psi \cdot \hat{\mathbf{n}}|_{\partial S \times (0,T)} = 0$,

$$\begin{aligned} &\int_S \left[\hat{\rho}\frac{|\hat{\mathbf{u}}|^2}{2} + \hat{\rho}\ln(\hat{\rho}) \right] \sqrt{d} \ d\hat{x} + \int_0^t \int_S P\left(|\omega(\hat{\mathbf{u}})|\right) |\omega(\hat{\mathbf{u}})|^2 \sqrt{d} \ d\hat{x}ds + \\ &+q \int_0^t \int_{\partial S} |\hat{\mathbf{u}}|^2 |R^{12}\hat{\mathbf{n}}| \sqrt{d} \ dSds + 2h \int_0^t \int_S |\hat{\mathbf{u}}|^2 \sqrt{d} \ d\hat{x}ds = \\ &= \int_0^t \int_S \widehat{\rho \mathbf{F}} \cdot \hat{\mathbf{u}} \sqrt{d} \ d\hat{x}ds + \int_S \frac{|(\rho \mathbf{u})_0|^2}{2\rho_0} \sqrt{d} \ d\hat{x} + \int_S \rho_0 \ln(\rho_0) \sqrt{d} \ d\hat{x}. \end{aligned} \tag{6.4.3}$$

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