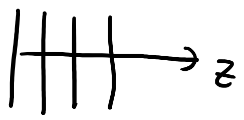
GAUSSIAN BEAMS

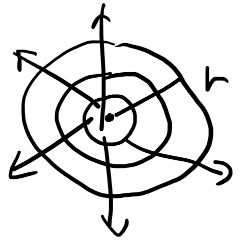
• $\boxed{\nabla^2 E - \frac{n^2}{c^2}\frac{\partial^2 E}{\partial t^2} = 0}$

$\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$, $\quad \frac{n^2}{c^2} = \frac{1}{v^2}$

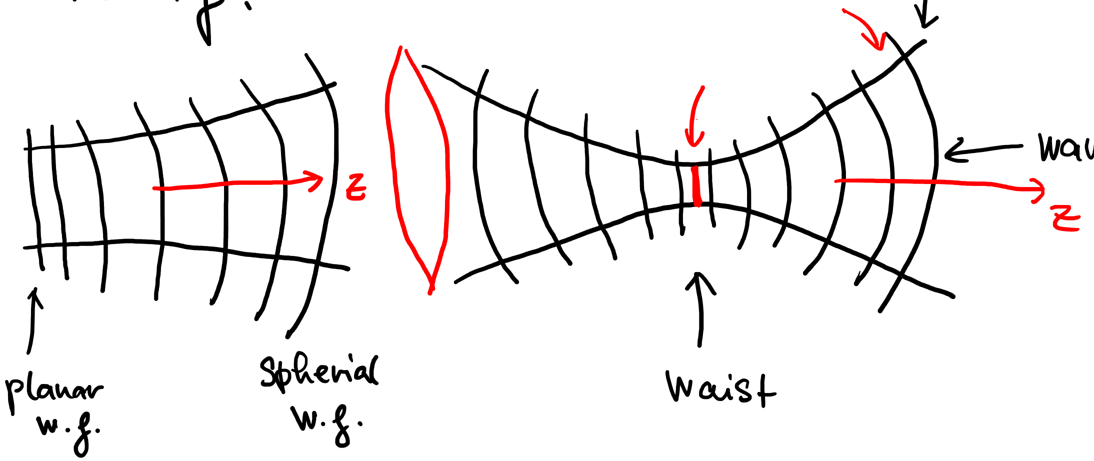
→ Plane wave $\quad E = E_0 e^{i(kz - \omega t + \phi)}$, $\quad kz = \vec{k}\cdot\vec{r}$, $\quad \vec{k} = (0,0,k)$



Spherical wave $\quad E = E_0 \frac{1}{r} e^{i(kr - \omega t + \phi)}$



→ in reality:



→ empirical solution: $\quad E(x,y,z,t) = \underset{1.}{U(x,y,z)}\, \underset{2.}{e^{i(kz - \omega t + \phi)}}$

1. complex ampl., varies slowly

2. fast oscillating plane wave

paraxial approx $\frac{\partial U}{\partial z} \ll k$

$$\to \quad e^{i(kz-\omega t+\phi)}\left[\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\underbrace{\frac{\partial^2 U}{\partial z^2}}_{=0}+2ik\frac{\partial U}{\partial z}-\underbrace{\left(k^2-n^2\frac{\omega^2}{c^2}\right)}_{=0}U\right]=0$$

$\frac{\partial^2 U}{\partial z^2} \ll 2ik\frac{\partial U}{\partial z}$

$k = \frac{n\omega}{c}$

$$\boxed{\underbrace{\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}}_{\nabla_T^2 U} + 2ki\frac{\partial U}{\partial z} = 0}$$ Paraxial Helmholtz eq.

$$\hookrightarrow \underbrace{U(x,y,z) = \frac{U_1}{z} e^{-ik\frac{\rho^2}{2z}}}_{\text{parabolic wave}} \qquad \rho^2 = x^2+y^2$$

$$z \to \boxed{z - \xi = q(z)}$$ shift in space; centered @ $z=\xi$ instead of $z=0$.

only imag.

$$\boxed{\xi = -i z_R} \quad z_R \in \mathbb{R}$$

$$\boxed{q(z) = z + i z_R} \in \mathbb{C}$$

$$\boxed{U(x,y,z) = \boxed{\frac{U_1}{q(z)}} e^{-ik\frac{\rho^2}{2q(z)}}}$$ Gaussian beam

we define:

$$\boxed{\frac{1}{q(z)} = \mathrm{Re}\left(\frac{1}{q}\right) + \mathrm{Im}\left(\frac{1}{q}\right) = \frac{1}{R(z)} - i\frac{\lambda}{\pi w^2(z)}}$$

$$\frac{1}{q} = \frac{1}{z+iz_R} = \frac{1}{z+iz_R}\cdot\frac{z-iz_R}{z-iz_R} = \frac{z-iz_R}{z^2+z_R^2} = \frac{1}{z\left(1+\left(\frac{z_R}{z}\right)^2\right)} - i\frac{1}{z_R\left(1+\left(\frac{z}{z_R}\right)^2\right)}$$

$(A+B)(A-B) = A^2-B^2$

$$\Rightarrow \quad w^2(z) = \dots \qquad R(z) = \dots \qquad z_R = \dots$$

GAUSSIAN BEAMS

$$\Rightarrow \quad E(x,y,z) = \boxed{E_0 \frac{W_0}{W(z)}} \exp\left[-\frac{\rho^2}{W^2(z)}\right] \exp\left[-ikz - ik\frac{\rho^2}{2R(z)} + i\psi\right] \exp\left[-i\omega t + i\phi\right] \quad \Leftarrow$$

$A(z)$

where:

$$W(z) = W_0 \sqrt{1 + \left(\frac{z}{z_R}\right)^2} \qquad E_0 = \frac{U_1}{i z_R} \qquad \rho = \sqrt{x^2 + y^2}$$

constant

$$R(z) = z\left[1 + \left(\frac{z_R}{z}\right)^2\right]$$

$$\psi(z) = \operatorname{arctg}\left(\frac{z}{z_R}\right)$$

waist

$$W_0 = \sqrt{\frac{\lambda z_R}{\pi}} \quad \leftrightarrow \quad \underline{z_R} = \frac{\pi W_0^2}{\underline{\lambda}}$$

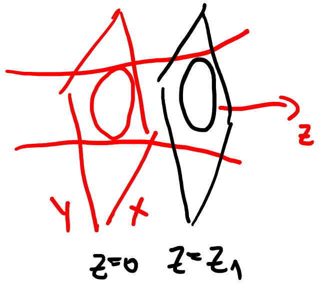
wavelength

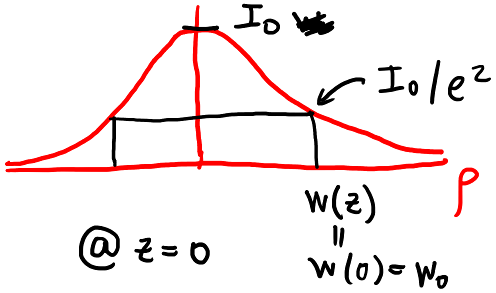
z_R, U_1, λ

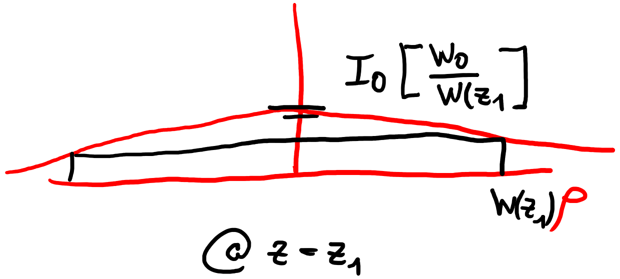
$q = z + i z_R \quad \rightarrow \quad \underline{q, U_1, \lambda}$

- **intensity:** $I(r) = |E|^2$: $I(\rho, z) = I_0 \left[\frac{W_0}{W(z)}\right]^2 \exp\left[-\frac{2\rho^2}{W^2(z)}\right]$ $\qquad I_0 = E_0^2$

$z = 0$







GAUSSIAN BEAMS

- Transmitted power: $P = \int_0^\infty I(\rho, z)\, 2\pi\rho \, d\rho = \frac{1}{2} I_0 (\pi W_0)^2 \neq f(z)$

- beam radius: at $\rho = W(z)$, I ~~drops~~ drops to I_0/e^2 → 86% is within $W(z)$

  ⇒ $W(z)$ is the beam width (radius)

$$W(z) = W_0 \sqrt{1 + (z/z_R)^2}$$

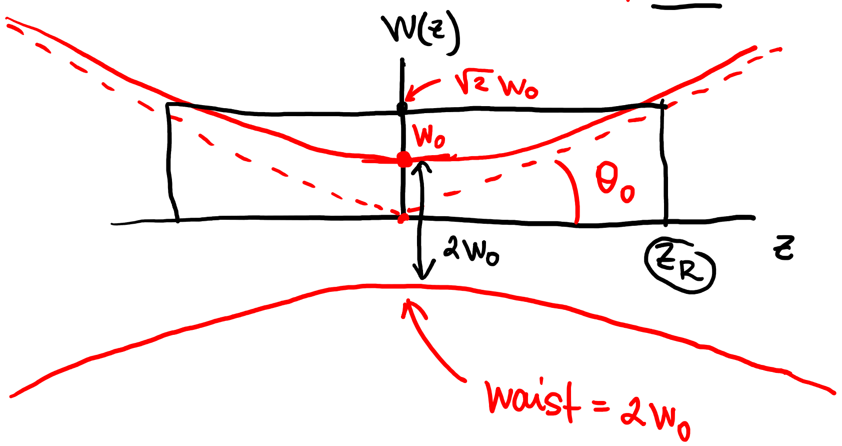
$z = 0$, $W(z) = W_0$

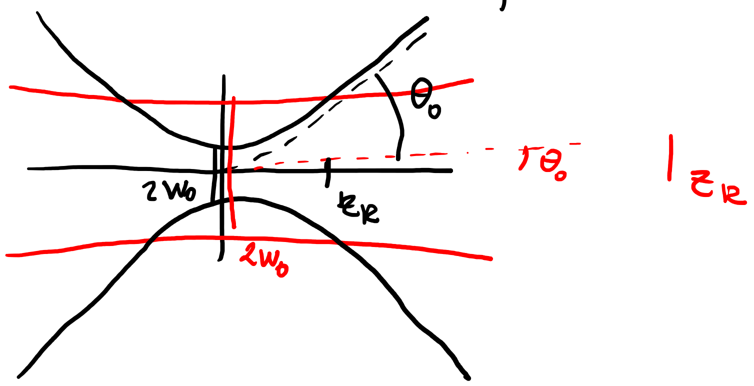
$z \gg z_R$; $\underline{W(z)} \cong W_0 \sqrt{(z/z_R)^2} = W_0 \frac{z}{z_R} = \theta_0 \underline{z}$

$\underline{\underline{\theta_0}} = \frac{W_0}{z_R}$ divergence angle

$z_R = \frac{\pi W_0^2}{\lambda}$

$W_0 = \sqrt{\frac{\lambda z_R}{\pi}}$





ex. $\lambda = 800\,nm$; $2W_0 = 1\,mm$; $2z_R = 2\,m$; $\theta_0 \cong 0°$

$= 10\,\mu m$; $2z_R = 200\,\mu m$; $\theta_0 \simeq 3°$

$= 1\,\mu m$; $= 2\,\mu m$; $\theta_0 \simeq 30°$

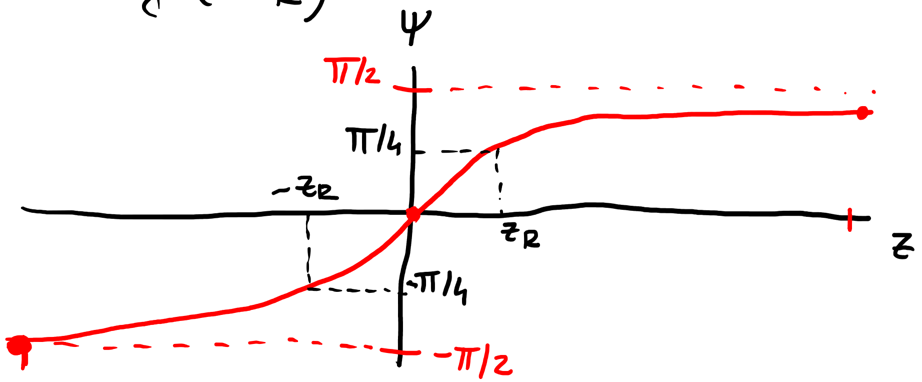
GAUSSIAN BEAMS

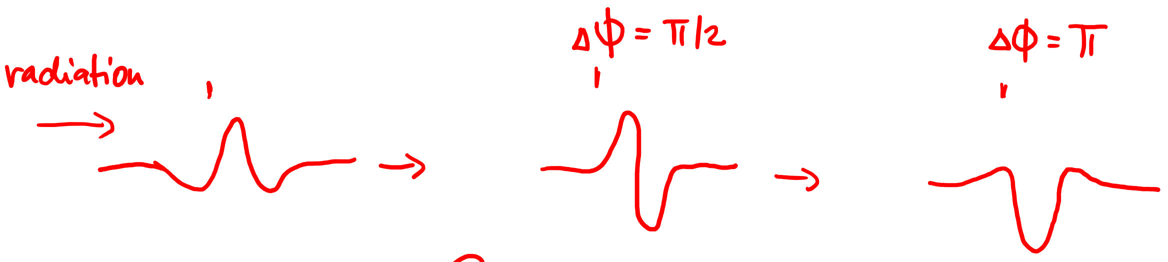
- phase : $-kz - k\frac{\rho^2}{2R(z)} + \psi \leftarrow \psi = \arctan(z/z_R)$

plane wave propagator

on the axis $\rho = 0$

$\Rightarrow = 0$





Guoy shift

- wave fronts : out of axis : $\rho \neq 0$ $\quad -k\frac{\rho^2}{2R(z)}$

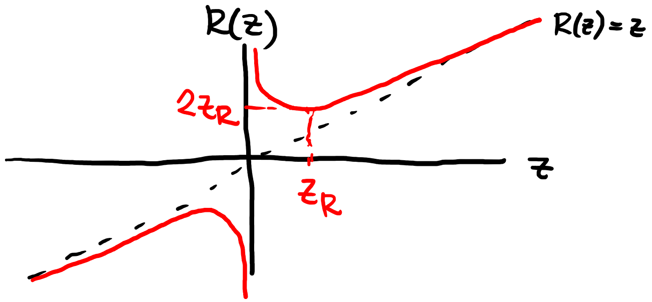
↳ w/ constant phase : $\text{const} = -kz - k\frac{\rho^2}{2R(z)} + \psi \leftarrow$ paraboloide w/ radius of curvature $R(z)$

$$R(z) = z\left[1 + \left(\frac{z_R}{z}\right)^2\right]$$

$z = 0$ .. $R(z) = \infty$

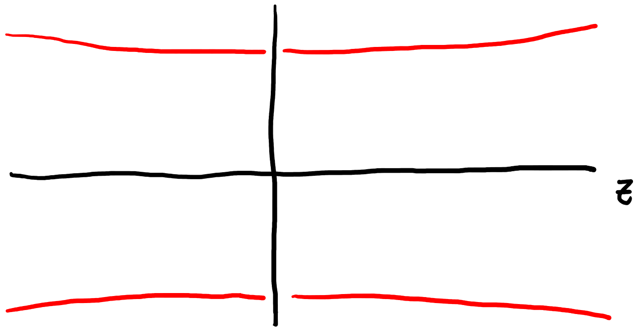
$z = z_R$ ... $R(z) = 2z_R$

$z \gg z_R$ ... $R(z) = z$

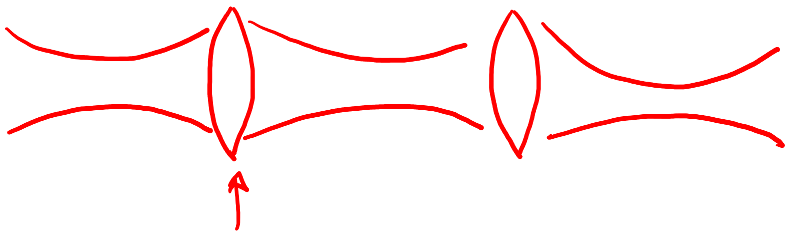


|| Transmission through optical elements ||

$$\boxed{\theta_0 = \frac{\lambda}{\pi w_0}}$$



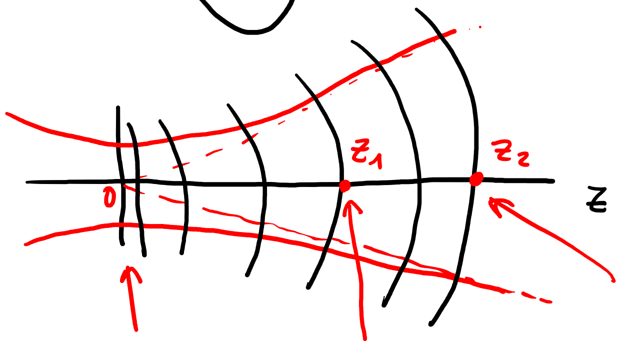
Lens propagator / lens waveguide



$q \longrightarrow$ q What is the meaning ?

• $\boxed{q = z + i z_R}$ $\quad z_R = \frac{\pi w_0^2}{\lambda}$

Free propagation



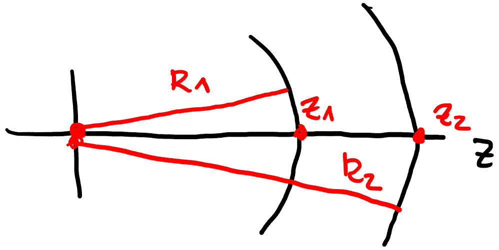
$q_0 = 0 + i z_R$
$= \text{number}$

$q_1 = z_1 + i z_R$

$q_2 = z_2 + i z_R$
$= q_1 + (z_2 - z_1)$

$\boxed{q \Leftrightarrow R}$

geometrical optics
spherical waves:



$R_1 = z_1$

$R_2 = z_2 = R_1 + (z_2 - z_1)$

$q \Longleftrightarrow R$ but $q \neq R$ $\quad \dfrac{1}{q(z)} = \dfrac{1}{R(z)} - i\dfrac{\lambda}{\pi w^2(z)}$

complex

$\dfrac{w^2}{\lambda}\pi$

$q \to R$?

$$R = \left[\mathrm{Re}\left\{ \frac{1}{q} \right\} \right]^{-1}$$

$q \to w$?

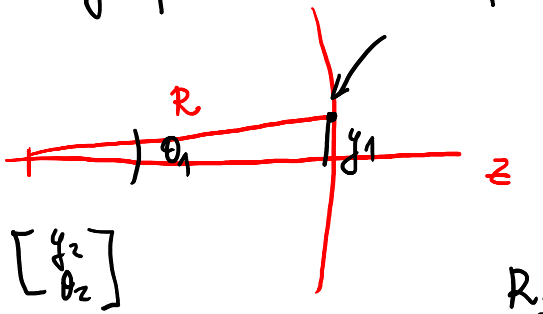
$$W = \left[\frac{\lambda}{\pi} \left[\mathrm{Im}\left\{ -\frac{1}{q} \right\} \right]^{-1} \right]^{1/2}$$

→ description of the transformation using "q": matrix optics, "ABCD" formalism

$$\begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix}$$

$$y_2 = A y_1 + B\theta_1$$

$$\theta_2 = C y_1 + D\theta_1$$



$\theta \approx \dfrac{y_1}{R} \Rightarrow R \approx \dfrac{y_1}{\theta_1}$

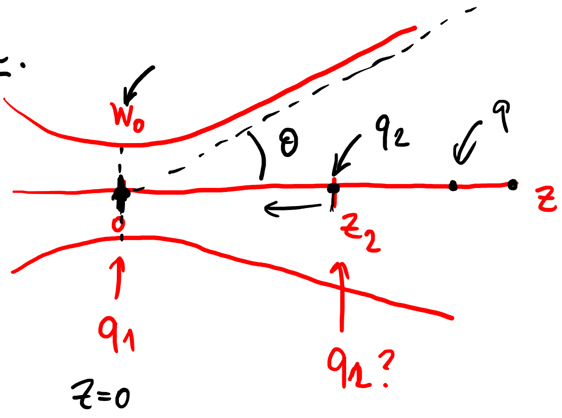
$$\begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix} \to \begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix}$$

$$R_1 \to R_2$$

$$R_2 = \frac{y_2}{\theta_2} = \frac{A y_1 + B\theta_1}{C y_1 + D\theta_1} = \frac{A\frac{y_1}{\theta_1} + B}{C\frac{y_1}{\theta_1} + D} = \frac{AR_1 + B}{CR_1 + D}$$

$$\boxed{q_2 = \frac{A q_1 + B}{C q_1 + D}} \qquad \boxed{\frac{1}{q_2} = \frac{C + D\frac{1}{q_1}}{A + B\frac{1}{q_1}}}$$

• ex.



$$M_{z_2} = \begin{bmatrix} 1 & z_2 \\ 0 & 1 \end{bmatrix} \Rightarrow \underline{\underline{q_2}} = \frac{1 \cdot q_1 + z_2}{0 \cdot q_1 + 1} = q_1 + \underline{\underline{z_2}} \quad \in \mathbb{R} \quad \checkmark$$

(?) waist : $\underline{\underline{W_0}} = \left[\mathrm{Im}\left\{ \frac{1}{q_2} \right\} \frac{\lambda}{\pi} \right]^{1/2} \neq f(\underline{z})$

(?) What is its location?

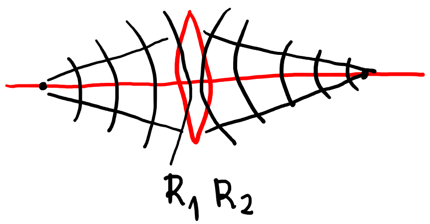
$$\boxed{z_W = -\,\mathrm{Re}\{q\}}$$

(?) R ? $\quad R = \left[\mathrm{Re}\left\{ \frac{1}{q} \right\} \right]^{-1}$

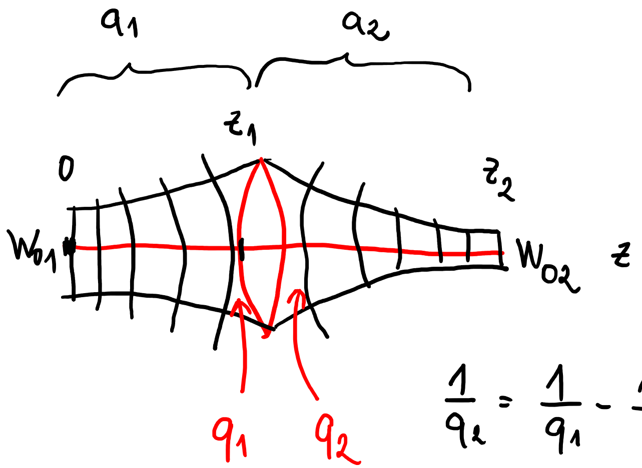
(?) Divergence : $\theta = \dfrac{\lambda}{\pi W_0}$

- lens:

in geometrical optics



$$\frac{1}{R_2} = \frac{1}{R_1} - \frac{1}{f}$$



$$\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{f} \quad ?$$

check: $M = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$

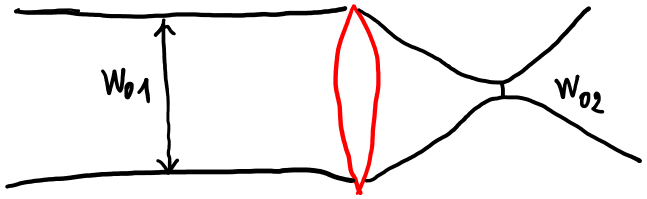
$$\frac{1}{q_2} = \frac{C + D\frac{1}{q_1}}{A + B\frac{1}{q_1}} = \frac{-\frac{1}{f} + \frac{1}{q_1}}{1 + 0\frac{1}{q_1}} = -\frac{1}{f} + \frac{1}{q_1} \quad \checkmark$$

$\to$ W_{02} : $\mathrm{Im}\left\{\frac{1}{q_2}\right\}$: $q_2 = \frac{f q_1}{-f + q_1}$ $\mathbb{C}$ $\mathbb{C}$ $\quad q_1 = z_1 + i z_R \quad \mathrm{Im}\{q_2\}$

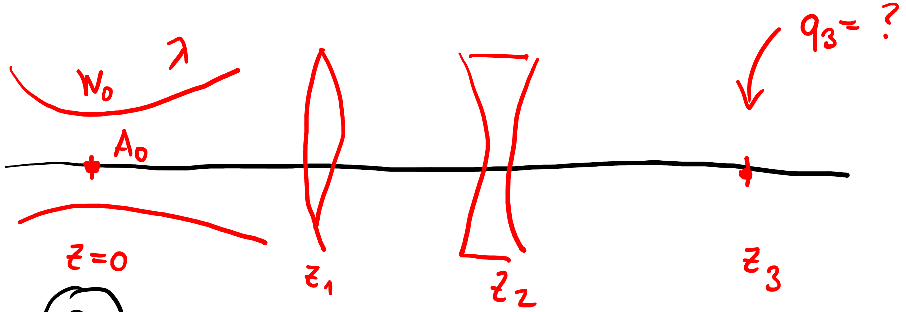
$\to$ $$\boxed{\begin{aligned} W_{02} &= W_{01} \cdot \frac{f}{\left[(z_1-f)^2 + z_{R1}^2\right]^{1/2}} \\ z_2 &= f \cdot \left[\frac{z_1(z_1-f) + z_{R1}^2}{(z_1-f)^2 + z_{R1}^2}\right] \end{aligned}}$$

$$z_1 = f \;\to\; W_{02} = W_{01}\frac{f}{z_{R1}} \;\to\; \boxed{W_{01} W_{02} = \frac{f\lambda}{\pi}}$$

Diffraction focusing limit



→



$$q_1 = 0 + i z_R$$

$$z_R = \frac{\pi w_0^2}{\lambda}$$

$$M = M_{z_3-z_2} M_{L2} M_{z_2-z_1} M_{L1} M_{z_1}$$

$$q_3 = \frac{A q_1 + B}{C q_1 + D}$$

Shape

- $$\boxed{\frac{A(z)}{A(0)} = \frac{\overset{q_1}{\overset{\parallel}{q(0)}}}{\underset{\underset{q_3}{\parallel}}{q(z)}}}$$