Clones, Constraints, and Minions

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POCOCOP

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OUTLINE

Part I

~100 years ago [Post]
Boolean clones

~50 years ago [Schaefer]
Boolean CSPs

~25 years ago [Feder, Vardi
Jeavons]
tight connection

~20 years ago [Bulatov, Jeavons, Krokhin]
abstract clones

~5 years ago [Barto, Opršal, Pinsker
B, Bulín, Krokhin, Opršal]
minions

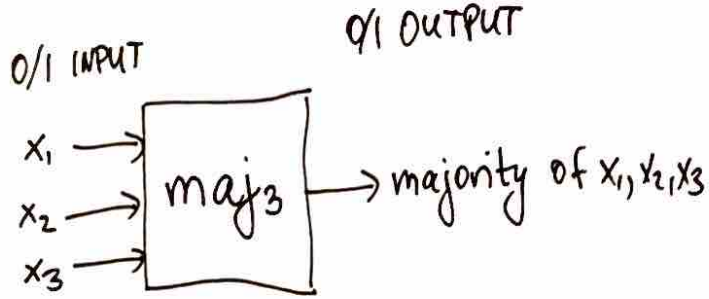
Part II

Recent work on clones on $\{0,1,2\}$

[B, Brady, Jankovec, Vucaj, Zhuk]

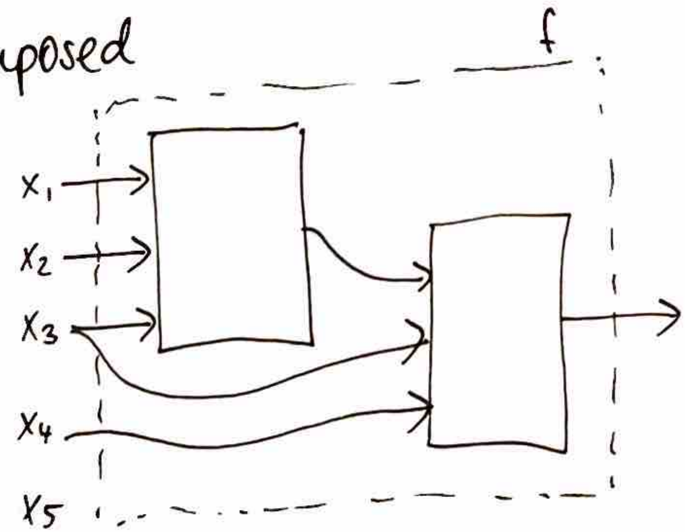
I. BOOLEAN CLONES: EXAMPLE

- consider



$$\{0,1\}^3 \longrightarrow \{0,1\}$$

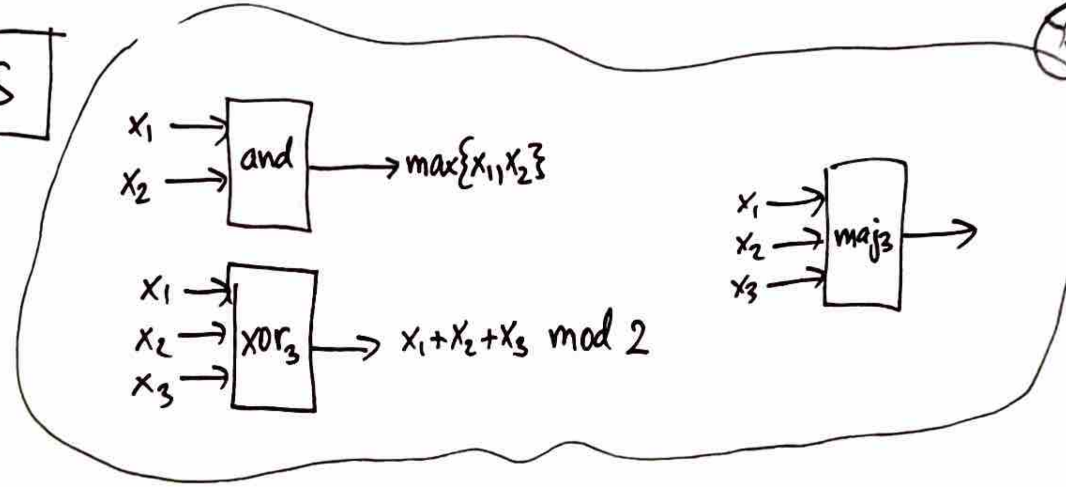
- can be composed



$$f(x_1, x_2, x_3, x_4, x_5) = maj_3(\, maj_3(x_1, x_2, x_3),\ x_3,\ x_4\,)$$

- – Can you compose maj_7 from maj_3? (maj_{37}, $maj_{1234567}$?)
  – what operations $\{0,1\}^n \to \{0,1\}$ can be composed from maj_3?

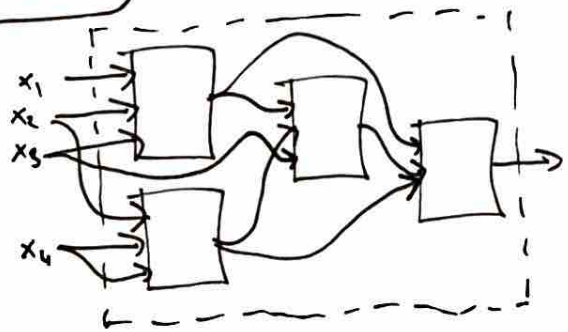
I. BOOLEAN CLONES : DEFINITIONS



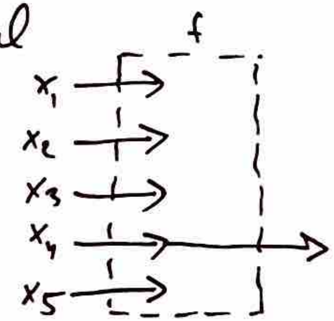
- Boolean operation $\{0,1\}^n \to \{0,1\}$

  (think: logical connective)

- composition



includes the trivial



$f(x_1, x_2, x_3, x_4, x_5) = x_4$

- Boolean clone set of Boolean operations closed under composition

- $\mathcal{F}$ set of Boolean operations

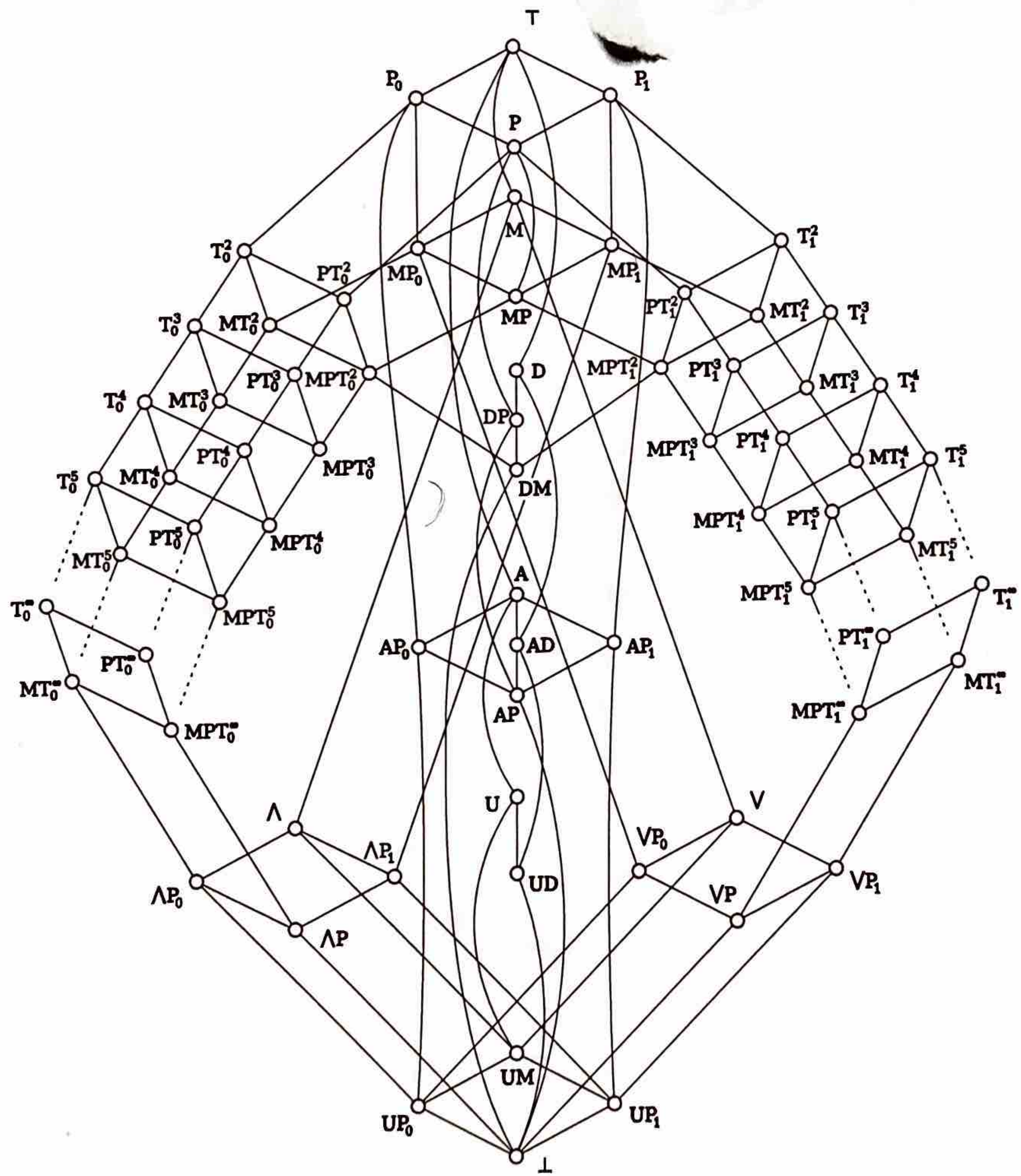
  $Clo(\mathcal{F})$ = the smallest clone containing $\mathcal{F}$

  - $Clo(\mathcal{F}) = Clo(\mathcal{G})$ ... $\mathcal{F}, \mathcal{G}$ equally expressive
  - $Clo(\mathcal{F}) \subset Clo(\mathcal{G})$ ... $\mathcal{G}$ more expressive than $\mathcal{F}$

last slide

- $maj_7 \in Clo(\{maj_3\})$?
- what is $Clo(\{maj_3\})$?

I. BOOLEAN CLONES – CLASSIFICATION



[Post '41 but...]

- poset
  (Boolean clones, $\subseteq$)

- = poset
  (sets of logical connectives,
  ~~$\leq$ expressive power~~)
  equal expressive power

I. GENERAL CLONES

- Clone on a set A

  set of operations on A ($A^n \to A$)
  closed under composition

- Dream: classify all clones

  More realistic: at least on $\{0,1,2\}$ ($2^{\aleph_0}$ many)

  More realistic: prove at least something

- Motivation
  - logic
  - algebra (algebras $\underline{A}, \underline{B}$ have the same clone of term operations $\Rightarrow$ they are "the same" for most purposes)
  - computational complexity...

I. BOOLEAN CSPs

Constraint Satisfaction Problems

- Boolean relation $R \subseteq \{0,1\}^n$

$R = \{0,1\}^2 \setminus \{(0,0)\}$ $R(x,y) \equiv x \vee y$

$S = \{0,1\}^3 \setminus \{(1,0,0)\}$ $S(x,y,z) \equiv \neg x \vee y \vee z$

- given Γ ... set of Boolean relations

$\Gamma = \{R, S\}$

CSP(Γ)

INPUT: pp-sentence over Γ

only $\exists$, $\wedge$, $R \in \Gamma$ allowed

OUTPUT: true?

$\exists x,y,z,u \; R(x,y) \wedge S(z,x,u) \wedge R(x,u)$

YES $x,y,z,u \mapsto 1$

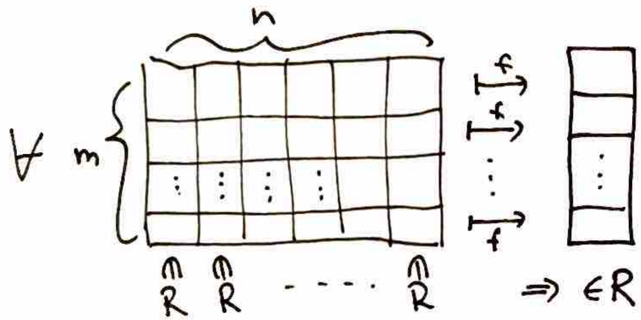
- Examples: 2-SAT, HORN-SAT, 3-SAT, 3-COLORING, LINEAR EQ. OVER GF(2)
- THEOREM [Schaefer '78] $\forall \Gamma$ CSP(Γ) < in P / NP-complete
- Main tool: Γ pp-definable from $\Delta \Rightarrow$ CSP(Γ) $\leq$ CSP(Δ)

I. CLONES $\leftrightarrow$ CSPs : HISTORY

- [Schaefer'78] Post did something about Boolean operations seems unrelated ...
- [Feder, Vardi'98] Complexity of CSP(Γ) seems related to „closure properties" of relations in Γ
- [Jeavons'00] Complexity of CSP(Γ) IS related to ——"—— (by a result from 60s) $\rightsquigarrow$ clones
- [universal algebraists] LET'S GO !!!

I. CLONES $\leftrightarrow$ CSPs : POLYMORPHISMS

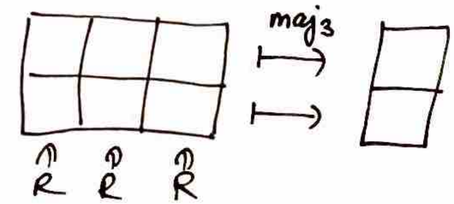
- $f : \{0,1\}^n \to \{0,1\}$ is a polymorphism of $R \subseteq \{0,1\}^m$ if



- Γ ... set of Boolean relations

Pol(Γ) ... polymorphisms of every $R \in \Gamma$. $\boxed{\text{Clone}}$

maj$_3$ is a polymorphism of every $R \subseteq \{0,1\}^2$



Pol(all binary) = Clo(maj$_3$)

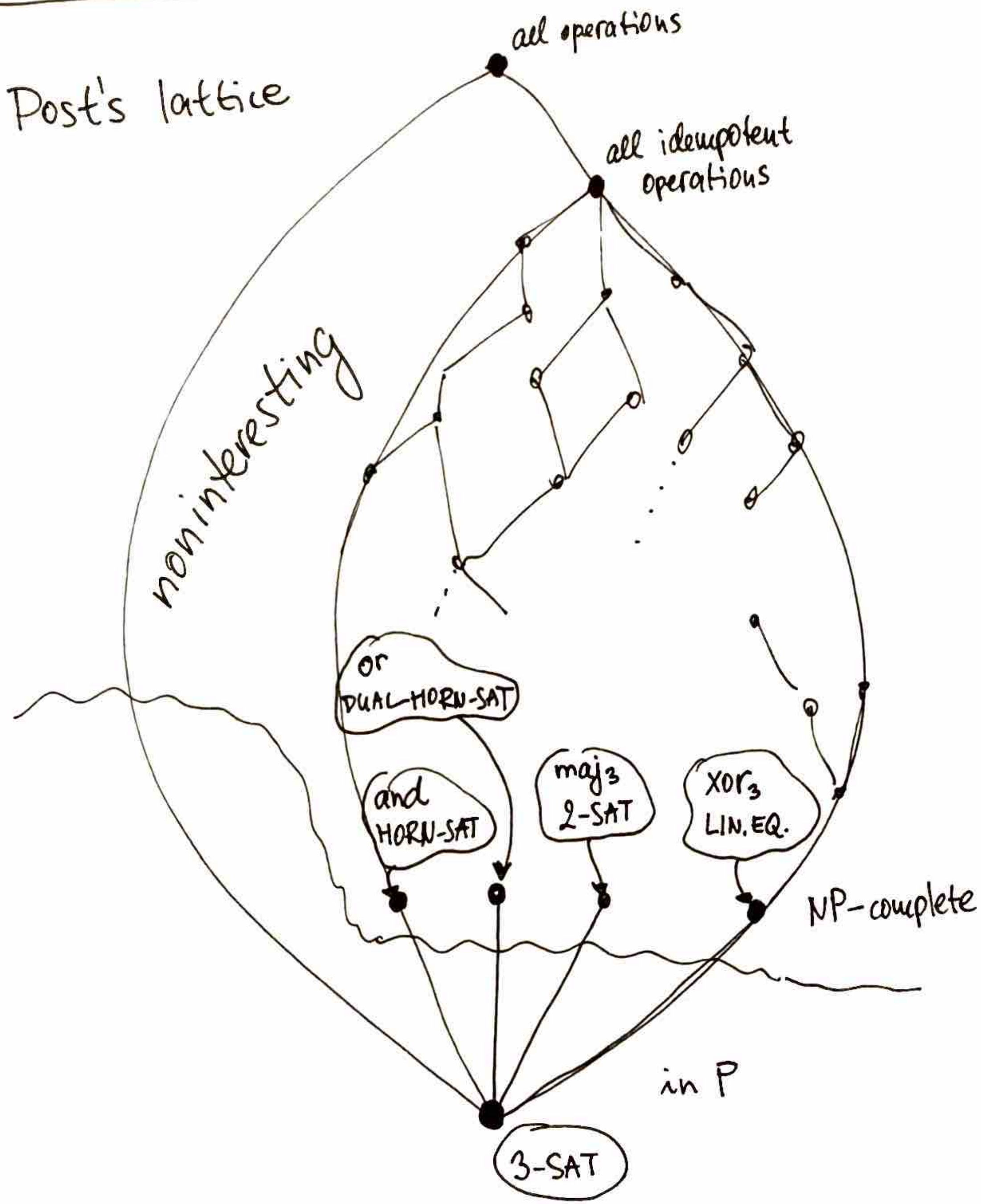
- THEOREM [Geiger'68, Bodnarchuk, Kaluzhnin, Kotov, Romov'69]

Γ pp-definable from $\Delta \iff$ Pol$(\Gamma) \supseteq$ Pol(Δ)

(and then CSP$(\Gamma) \leq$ CSP(Δ))
[Schaefer]

not only Boolean
... any finite A

I. CLONES ⟷ CSPs: SCHAEFER REVISITED



- 60s theorem implies

  Pol(Γ) higher

  ⇒ CSP(Γ) easier

- restrict to "interesting clones"

  = idempotent, i.e. $\forall a \; f(a,a,\dots,a)=a$

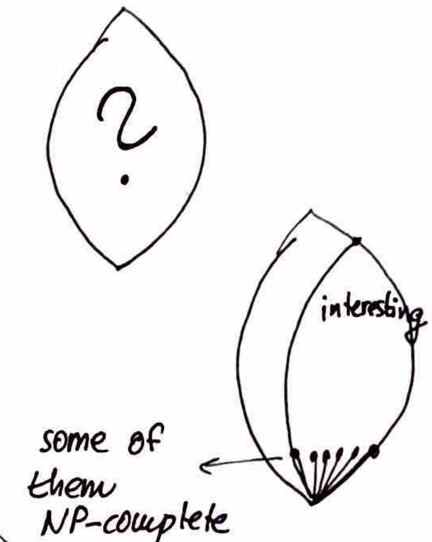
- 4 minimal clones by Post
  - Clo(and)
  - Clo(or)
  - Clo(maj_3)
  - Clo(xor_3)

- they correspond to CSPs in P

  ⇒ Schaefer

I. NEW ORDERS

clones on $\{0,1,2\}$

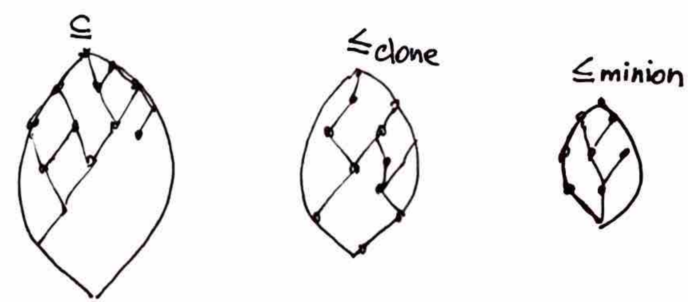


- 2 problems beyond Boolean
  1. classification hard even for clones on $\{0,1,2\}$
  2. for CSPs: borderline is not at the bottom

- "better preorders" $\mathcal{C} \subseteq \mathcal{D} \Rightarrow \mathcal{C} \leq_{clone} \mathcal{D} \Rightarrow \mathcal{C} \leq_{minion} \mathcal{D}$

  ⇝ "simpler" posets

  still good for CSPs



⇝ "simpler" objects than clones

(here "simpler" = carries less information)

I. NEW ORDERS: DETAILS

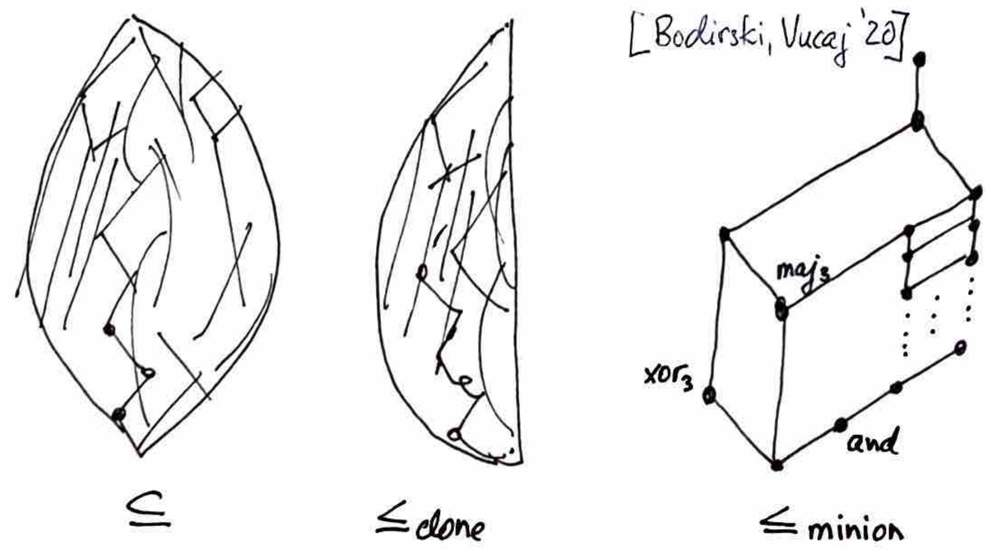
(my life ① minions
② algebras, clones, CSPs
③ minions)

$\mathrm{CSP}(\Gamma) \leq \mathrm{CSP}(\Delta)$ if

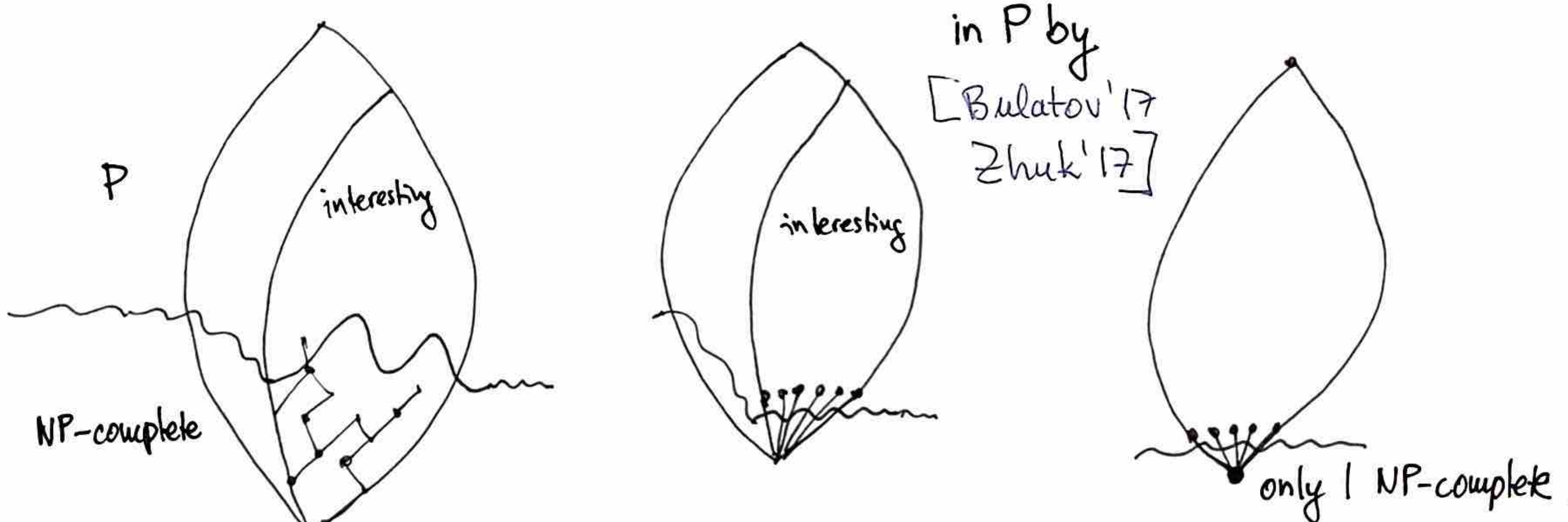
relationally	Γ pp-definable from Δ	Γ pp-interpretable in Δ	Γ pp-constructible from Δ
algebraically	$\mathrm{Pol}(\Delta) \subseteq \mathrm{Pol}(\Gamma)$	$\exists\, \mathrm{Pol}(\Delta) \to \mathrm{Pol}(\Gamma)$ clone homomorphism preserves arities & composition	$\exists\, \mathrm{Pol}(\Delta) \to \mathrm{Pol}(\Gamma)$ minion homomorphism preserves arities & permutation, identification of variables
preorder on clones	$\mathcal{C} \subseteq \mathcal{D}$	$\mathcal{C} \leq_{clone} \mathcal{D}$ if $\exists$ clone homo $\mathcal{C} \to \mathcal{D}$	$\mathcal{C} \leq_{minion} \mathcal{D}$ if $\exists$ minion homo $\mathcal{C} \to \mathcal{D}$
algebraic object	clone [Geiger'68; Bodnarchuk, Kaluzhnin Kotov, Romov'69 Jeavons'00]	abstract clone [Birkhoff '35 Bulatov, Jeavons, Krokhin'05 Bodirsky'08]	minion functor finite ordinals $\to$ sets [B, Opršal, Pinsker'18 B, Bulín, Krokhin, Opršal'21] * see my life

I. NEW ORDERS : GAINS

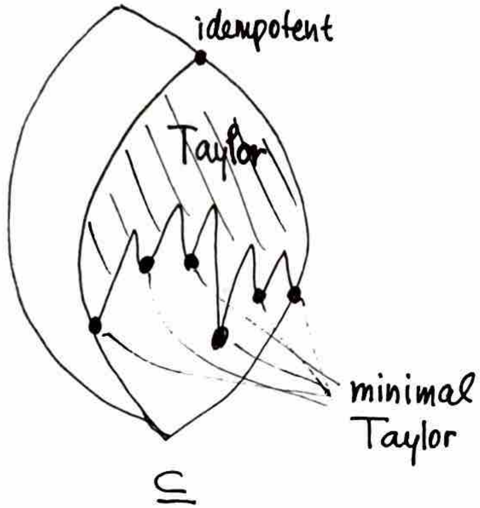
- Boolean clones <u>get</u> simpler



- new hope for $\{0,1,2\}$ (©Zhuk) or even bigger, e.g. possibly countable!
- beautiful borderlines for CSPs



II. MINIMAL TAYLOR CLONES: DEFINITION



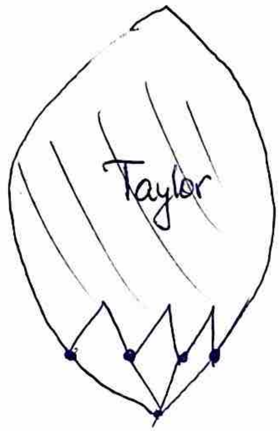
- **Taylor clone** – idempotent (Remark: $\forall \mathcal{C}\ \exists$ idempotent $\mathcal{D}$ $\ \mathcal{C} \sim_{minion} \mathcal{D}$)
  – not at the bottom wrt. $\leq_{minion}$ ($\Leftrightarrow$ $\leq_{clone}$)

- **Minimal Taylor clone on A** minimal such wrt. $\subseteq$

- Why? – algebraic interest (smallest "equationally nontrivial" clones)
  – CSP interest (hardest CSPs in P)

II. MINIMAL TAYLOR CLONES: $A = \{0,1\}$

[Post'41]



- 4 minimal Taylor clones
  - $\mathrm{Clo}(\{\mathrm{and}\})$
  - $\mathrm{Clo}(\{\mathrm{or}\})$
  - $\mathrm{Clo}(\{\mathrm{maj}_3\})$
  - $\mathrm{Clo}(\{\mathrm{xor}_3\})$

they are minimal

- 3 up to renaming $0 \leftrightarrow 1$

- every Taylor clone on $\{0,1\}$ contains a minimal Taylor clone

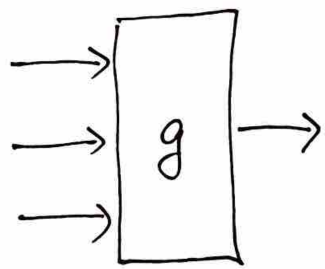
- every minimal Taylor clone $\mathcal{C}$ on $\{0,1\}$
  is $\mathcal{C} = \mathrm{Clo}(\{g\})$
  for some $g : \{0,1\}^3 \to \{0,1\}$

II. MINIMAL TAYLOR CLONES: FUN FACTS

[B, Brady, Bulatov, Kozik, Zhuk'24]

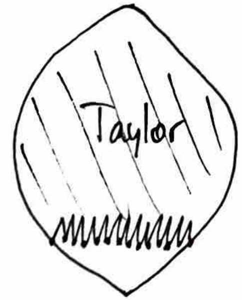
- every Taylor clone on a finite A contains a minimal Taylor clone

- every minimal Taylor clone $\mathcal{C}$ on a finite A is $\mathcal{C} = \mathrm{Clo}(\{g\})$ for some $g : A^3 \to A$



II. MINIMAL TAYLOR CLONES: $A = \{0,1,2\}$

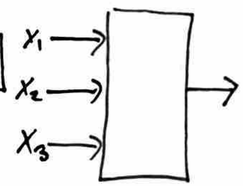
[B, Brady, Jankovec, Vucaj, Zhuk 'TBD]



- all minimal Taylor clones on $\{0,1,2\}$
  - 95
  - 24 up to renaming elements
- all minimal Taylor **conservative** clones on any finite A

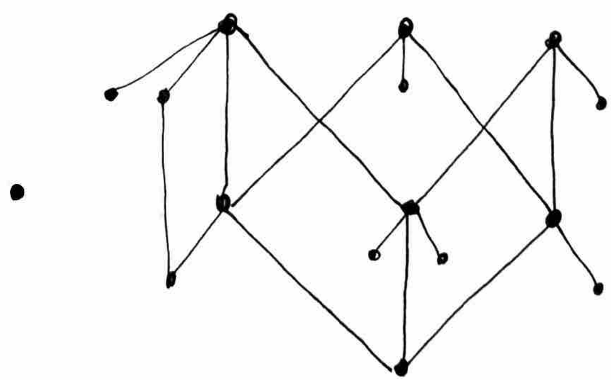
  $\forall f \ \forall x_1,\dots,x_n \ f(x_1,\dots,x_n) \in \{x_1,\dots,x_n\}$
- ↑ described the generating $g: \{0,1,2\}^3 \to \{0,1,2\}$.
  $Clo(\{g\})$ desribed for 18/24

$\boxed{\text{open}}$



$$\begin{cases} maj_3(x_1,x_2,x_3) & \text{if } \{x_1,x_2,x_3\} \subseteq \{0,1\} \text{ or } \{x_1,x_2,x_3\} \subseteq \{1,2\} \\ 2 & \text{else} \end{cases}$$

- their $\leq_1$ order (hopefully)



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III. WRAP UP

- clones $\longrightarrow$ abstract clones $\longrightarrow$ (abstract) minions $\longrightarrow$?
- minions important for Promise CSPs

  e.g. 6-color a 3-colorable graph
- math: very infinite, infinite, finite, very finite

  active & cool part of logic & algebra
- object $G \longrightarrow \mathrm{Aut}(G)$ group: symmetries of G

  $G \longrightarrow \mathrm{Pol}(G)$ clone: multivariate symmetries of G

  - useful in CSPs
  - will it become a standard concept in math?

Thank you for listening!