Vanishing negative K-theory and bounded t-structures

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4 March 2025

Overview

1. Grothendieck's K_0
2. Higher K–theory
3. The general K-theoretic conjectures
4. The counterexample
5. The scheme-theoretic conjecture and its proof

Exact categories

Definition

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Example

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2.
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2. A full subcategory $\mathcal{E} \subset \mathcal{A}$, closed under extensions and with $\mathcal{A}$ abelian.
3. $\mathrm{Vect}(X)$, the category of vector bundles over X.

Exact categories

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Given any additive category $\mathcal{E}$, we can turn it into an exact category by declaring the sequences

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to be the admissible exact sequences. We will write $\mathcal{E}^{\oplus}$ for this exact category.

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Remark

Let X be a scheme and put $\mathcal{E} = \mathrm{Vect}(X)$.
Then $\mathcal{E} = \mathcal{E}^{\oplus}$ when X is affine, but not otherwise.

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Definition (convenient shorthand)

When X is a reasonable scheme, we define $K_0(X) = K_0\big[\mathrm{Vect}(X)\big]$.

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Higher K–theory

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there are obvious restriction functors

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Mayer-Vietoris sequence, continued

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$$\longrightarrow K_{-1}(X) \longrightarrow K_{-1}(U) \oplus K_{-1}(V) \longrightarrow K_{-1}(U \cap V) \longrightarrow \cdots$$

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This turns out to be possible. It is the culmination of the work of many people.

Vanishing observations

Two conjectures

1. **Weibel's conjecture:** If X is a noetherian scheme of dimension n, then $K_r(X) = 0$ for all $r < -n$.

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2. **Schlichting's theorem:** It is a theorem that, if X is a noetherian, regular and finite dimensional scheme, then $K_r(X) = 0$ for all $r < 0$.

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   Schlichting conjectured a major generalization.

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Focus on Schlichting's conjecture

Schlichting conjecture isn't only about schemes.

Remember: given any exact category $\mathcal{E}$ there is a recipe to produce a K-theory out of it. And until now we have focused on the case $\mathcal{E} = \mathrm{Vect}(X)$.

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(Conjecture A)

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Schlichting proved:

1. If the abelian category $\mathcal{A}$ is noetherian, then $K_n(\mathcal{A}) = 0$ for $n < 0$.
2. For any abelian category $\mathcal{A}$, we have $K_{-1}(\mathcal{A}) = 0$.

Plausibility argument

Theorem (Quillen). Suppose $\mathcal{B}$ is an abelian category, assume $\mathcal{A} \subset \mathcal{B}$ is a Serre subcategory, and let $\mathcal{C} = \mathcal{B}/\mathcal{A}$.

$$\begin{array}{ccccc} K_1(\mathcal{A}) & \longrightarrow & K_1(\mathcal{B}) & \longrightarrow & K_1(\mathcal{C}) \\ K_0(\mathcal{A}) & \longrightarrow & K_0(\mathcal{B}) & \longrightarrow & K_0(\mathcal{C}) \\ 0 & & & & \end{array}$$

Daniel Quillen, *Higher algebraic K-theory. I*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Lecture Notes in Math., vol. 341, Springer verlag, 1973, pp. 85–147.

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How difficult can it be?

Given $\mathcal{A}$ we want to construct

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The reason is: we can form $F : \mathcal{B} \longrightarrow \mathcal{B}$ by the formula

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$$F(B) \cong B \oplus F(B) \qquad \text{hence} \qquad K_n(F) = K_n(\mathrm{id}) + K_n(F)$$

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The image of a map

$$\coprod_{i=1}^{\infty} A_i \longrightarrow A$$

will not usually lie in $\mathcal{A}$.

Let $\mathcal{T}$ be a model category with a bounded t–structure. Antieau, Gepner and Heller proved the following generalization of Schlichting's results:

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Benjamin Antieau, David Gepner, and Jeremiah Heller, *K-theoretic obstructions to bounded t-structures*, Invent. Math. **216** (2019), no. 1, 241–300.

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If $\mathcal{A}$ is an abelian category, Schlichting's results come about by putting $\mathcal{T} = \mathbf{D}^b(\mathcal{A})$ with the standard t–structure.

Benjamin Antieau, David Gepner, and Jeremiah Heller, *K-theoretic obstructions to bounded t-structures*, Invent. Math. **216** (2019), no. 1, 241–300.

The generalized Schlichting conjecture

For any $\mathcal{T}$ with a bounded t–structure, $K_n(\mathcal{T}) = 0$ for all $n < 0$.

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Yet another conjecture, in case the above are false

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Plausibility argument

Let $\mathcal{R} \subset \mathcal{S}$ be model categories with $\mathcal{T} = \mathcal{S}/\mathcal{R}$. Then

$$
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 & & & & \swarrow \\
K_{-1}(\mathcal{R}) & \longrightarrow & K_{-1}(\mathcal{S}) & \longrightarrow & K_{-1}(\mathcal{T}) \\
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K_{-2}(\mathcal{R}) & \longrightarrow & K_{-2}(\mathcal{S}) & \longrightarrow & K_{-2}(\mathcal{T}) \\
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Punchline

Schlichting's conjecture (Conjecture A)

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The counterexample appeared in

Amnon Neeman, *A counterexample to vanishing conjectures for negative K-theory*, Invent. Math. **225** (2021), no. 2, 427–452.

The categories $\mathbf{Ac}^b(\mathcal{E}) \subset \mathbf{K}^b(\mathcal{E})$ and $\mathbf{D}^b(\mathcal{E}) = \mathbf{K}^b(\mathcal{E})/\mathbf{Ac}^b(\mathcal{E})$

Let $\mathcal{E}$ be any idempotent-complete exact category. Let $\mathbf{K}^b(\mathcal{E})$ be the category whose objects are bounded cochain complexes in $\mathcal{E}$, meaning

$$\cdots \xrightarrow{\partial^{i-2}} E^{i-1} \xrightarrow{\partial^{i-1}} E^i \xrightarrow{\partial^i} E^{i+1} \xrightarrow{\partial^{i+1}} \cdots$$

with $E^i = 0$ for $|i| \gg 0$.

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The full subcategory $\mathbf{Ac}^b(\mathcal{E})$ of acyclics contains those cochain complexes for which there exist admissible short exact sequences

$$0 \longrightarrow K^i \xrightarrow{\alpha^i} E^i \xrightarrow{\beta^i} K^{i+1} \longrightarrow 0$$

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The t–structure on $\mathbf{Ac}^b(\mathcal{E})$

$$\begin{aligned} \mathbf{Ac}^b(\mathcal{E})^{\leq 0} &= \{E^* \in \mathbf{Ac}^b(\mathcal{E}) \mid E^i = 0 \text{ for all } i > 0\} \\ \mathbf{Ac}^b(\mathcal{E})^{\geq 0} &= \{E^* \in \mathbf{Ac}^b(\mathcal{E}) \mid E^i = 0 \text{ for all } i < -2\} \end{aligned}$$

Proof that this is a t–structure

A morphism from an object $E^* \in \mathbf{Ac}^b(\mathcal{E})^{\leq 0}$ to an object $F^* \in \mathbf{Ac}^b(\mathcal{E})^{\geq 1}$ may be represented by a cochain map

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\partial^{-3}} & E^{-2} & \xrightarrow{\partial^{-2}} & E^{-1} & \xrightarrow{\partial^{-1}} & E^{0} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & F^{-1} & \xrightarrow[\widetilde{\partial}^{-1}]{} & F^{0} & \xrightarrow[\widetilde{\partial}^{0}]{} & F^{1} & \xrightarrow[\widetilde{\partial}^{1}]{} & \cdots
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\end{array}$$

Proof that this is a t–structure

A morphism from an object $E^* \in \mathbf{Ac}^b(\mathcal{E})^{\leq 0}$ to an object $F^* \in \mathbf{Ac}^b(\mathcal{E})^{\geq 1}$ may be represented by a cochain map

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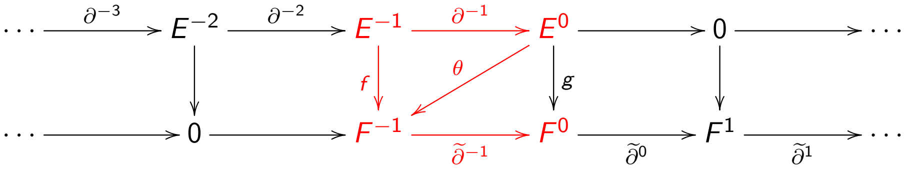

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Proof that this is a t–structure, continued

Next choose any object $E^* \in \mathbf{Ac}^b(\mathcal{E})$, that is a complex

$$\cdots \xrightarrow{\partial^{i-2}} E^{i-1} \xrightarrow{\partial^{i-1}} E^{i} \xrightarrow{\partial^{i}} E^{i+1} \xrightarrow{\partial^{i+1}} \cdots$$

Proof that this is a t–structure, continued

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Write $\partial^{-1} : E^{-1} \longrightarrow E^0$ as a composite $E^{-1} \xrightarrow{\beta^{-1}} K^0 \xrightarrow{\alpha^0} E^0$.

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Write $\partial^{-1} : E^{-1} \longrightarrow E^0$ as a composite $E^{-1} \stackrel{\beta^{-1}}{\longrightarrow} K^0 \stackrel{\alpha^0}{\longrightarrow} E^0$. Now consider the cochain maps

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\ \partial^{-3}\ } & E^{-2} & \xrightarrow{\ \partial^{-2}\ } & E^{-1} & \xrightarrow{\ \beta^{-1}\ } & K^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \alpha^0} & & \downarrow & & \\
\cdots & \xrightarrow{\ \partial^{-3}\ } & E^{-2} & \xrightarrow{\ \partial^{-2}\ } & E^{-1} & \xrightarrow{\ \partial^{-1}\ } & E^0 & \xrightarrow{\ \partial^{0}\ } & E^1 & \xrightarrow{\ \partial^{1}\ } & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle \beta^{-1}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & K^0 & \xrightarrow{\ \alpha^{0}\ } & E^0 & \xrightarrow{\ \partial^{0}\ } & E^1 & \xrightarrow{\ \partial^{1}\ } & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \\
\cdots & \xrightarrow{\ -\partial^{-2}\ } & E^{-1} & \xrightarrow{\ -\beta^{-1}\ } & K^0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

The heart

The heart of this t–structure, denoted $\mathbf{Ac}^b(\mathcal{E})^\heartsuit$, is by definition the full subcategory

$$\mathbf{Ac}^b(\mathcal{E})^\heartsuit \quad = \quad \mathbf{Ac}^b(\mathcal{E})^{\leq 0} \cap \mathbf{Ac}^b(\mathcal{E})^{\geq 0} \ .$$

The objects are the acyclic cochain complexes

$$0 \longrightarrow E^{-2} \longrightarrow E^{-1} \longrightarrow E^0 \longrightarrow 0$$

and the morphisms are the homotopy equivalence classes of cochain maps.

Formal consequence of the general theory

The category $\mathbf{Ac}^b(\mathcal{E})^\heartsuit$ is abelian.

Now we have $\mathbf{Ac}^b(\mathcal{E}) \subset \mathbf{K}^b(\mathcal{E})$ with quotient $\mathbf{D}^b(\mathcal{E})$, giving

$$\begin{array}{ccccc}
 & & & & \swarrow \\
K_{-1}(\mathbf{Ac}^b(\mathcal{E})) & \longrightarrow & K_{-1}(\mathbf{K}^b(\mathcal{E})) & \longrightarrow & K_{-1}(\mathbf{D}^b(\mathcal{E})) \\
 & & \swarrow & & \\
K_{-2}(\mathbf{Ac}^b(\mathcal{E})) & \longrightarrow & K_{-2}(\mathbf{K}^b(\mathcal{E})) & \longrightarrow & K_{-2}(\mathbf{D}^b(\mathcal{E})) \\
 & & \swarrow & & \\
K_{-3}(\mathbf{Ac}^b(\mathcal{E})) & \longrightarrow & K_{-3}(\mathbf{K}^b(\mathcal{E})) & \longrightarrow & K_{-3}(\mathbf{D}^b(\mathcal{E})) \\
 & & \swarrow & &
\end{array}$$

which rewrites as

$$
\begin{array}{ccccc}
 & & & & \swarrow \\
K_{-1}(\mathbf{Ac}^b(\mathcal{E})) & \longrightarrow & K_{-1}(\mathcal{E}^{\oplus}) & \longrightarrow & K_{-1}(\mathcal{E}) \\
 & \swarrow & & & \\
K_{-2}(\mathbf{Ac}^b(\mathcal{E})) & \longrightarrow & K_{-2}(\mathcal{E}^{\oplus}) & \longrightarrow & K_{-2}(\mathcal{E}) \\
 & \swarrow & & & \\
K_{-3}(\mathbf{Ac}^b(\mathcal{E})) & \longrightarrow & K_{-3}(\mathcal{E}^{\oplus}) & \longrightarrow & K_{-3}(\mathcal{E}) \\
 \swarrow & & & &
\end{array}
$$

Thus the vanishing of $K_n(\mathbf{Ac}^b(\mathcal{E}))$ for all $n < 0$ would imply that the map

$$K_n(\mathcal{E}^\oplus) \longrightarrow K_n(\mathcal{E})$$

would have to be an isomorphism for all $n < 0$. Hence, for a counterexample to the generalized Schlichting conjecture, all we need to do is find an $\mathcal{E}$ for which this fails.

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If we want to disprove the (ungeneralized) Schlichting conjecture and/or to study the yet another conjecture, then it might be helpful to look at the natural map

$$K_n\Big(\mathbf{Ac}^b(\mathcal{E})^{\heartsuit}\Big) \longrightarrow K_n(\mathbf{Ac}^b(\mathcal{E})) .$$

Theorem

Let $\mathcal{E}$ be an idempotent-complete exact category. Then the natural functor

$$D^b\Big(\mathbf{Ac}^b(\mathcal{E})^\heartsuit\Big) \longrightarrow \mathbf{Ac}^b(\mathcal{E})$$

is an equivalence of triangulated categories if and only if $\mathcal{E}$ is hereditary, meaning $\mathrm{Ext}^i(E,E')=0$ for all $i>1$ and $E,E'\in\mathcal{E}$.

Corollary

Theorem

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Corollary

If $\mathcal{E}$ is hereditary then the map

$$K_n\left(\mathbf{Ac}^b(\mathcal{E})^\heartsuit\right) \longrightarrow K_n(\mathbf{Ac}^b(\mathcal{E}))$$

must be an isomorphism for all $n \in \mathbb{Z}$.

Example

If Y is any algebraic curve, then the category $\mathcal{E} = \mathrm{Vect}(Y)$ is hereditary.

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After all: there is a spectral sequence

$$H^i(\mathcal{E}xt^j(E, E')) \Longrightarrow Ext^{i+j}(E, E'),$$

For vector bundles we know the vanishing of $\mathcal{E}xt^j(E, E')$ for $j > 0$, and for curves we have the vanishing of H^i for $i > 1$.

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For vector bundles we know the vanishing of $\mathcal{E}xt^j(E, E')$ for $j > 0$, and for curves we have the vanishing of H^i for $i > 1$.

The corollary on the previous page informs us that, for any algebraic curve Y and with $\mathcal{E} = \mathrm{Vect}(Y)$, the natural map

$$K_n\left(\mathbf{Ac}^b(\mathcal{E})^{\heartsuit}\right) \longrightarrow K_n(\mathbf{Ac}^b(\mathcal{E}))$$

is an isomorphism for all $n \in \mathbb{Z}$.

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I specialize to the case of singular projective curves with only simple nodes as singularities, directly prove that $K_{-1}(\mathcal{E}^{\oplus}) = 0$, and then cite the known examples where $K_{-1}(\mathcal{E}) \neq 0$.

Recall the general exact sequence

$$\begin{array}{ccccc} & & & & \swarrow \\ K_{-1}(\mathbf{Ac}^b(\mathcal{E})) & \longrightarrow & K_{-1}(\mathcal{E}^{\oplus}) & \longrightarrow & K_{-1}(\mathcal{E}) \\ & & & \swarrow & \\ K_{-2}(\mathbf{Ac}^b(\mathcal{E})) & \longrightarrow & K_{-2}(\mathcal{E}^{\oplus}) & \longrightarrow & K_{-2}(\mathcal{E}) \\ & & & \swarrow & \\ K_{-3}(\mathbf{Ac}^b(\mathcal{E})) & \longrightarrow & K_{-3}(\mathcal{E}^{\oplus}) & \longrightarrow & K_{-3}(\mathcal{E}) \\ & & & \swarrow & \end{array}$$

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 & & \swarrow & & \\
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LOUSY ARGUMENT!

The right approach would have been to prove the more general statement:

Theorem

Let $\mathcal{E}$ be an idempotent-complete additive category. Assume that, for all objects $E \in \mathcal{E}$, the ring $\mathrm{Hom}(E, E)$ is Artinian.

Then $K_n(\mathcal{E}^{\oplus}) = 0$ for all $n < 0$.

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$$
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 & \swarrow & & & \\
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If $\mathcal{E}$ has Artinian endomorphism rings

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If $\mathcal{E}$ has Artinian endomorphism rings

For a discussion of how this might lead to counterexamples to Conjecture C, the reader is referred to

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`arXiv:2208.06863`.

By now we know that Conjecture C is definitely false, see the preprint

M. Ramzi, V. Sosnilo, and C. Winges.
Every spectrum is the K-theory of a stable ∞-category.
arXiv:2401.06510.

The rough sketch goes as follows: in the counterexample I explained, we start with an exact category $\mathcal{E}$, and out of it construct a new category $\mathbf{Ac}^b(\mathcal{E})$, and the key properties are:

1. The category $\mathbf{Ac}^b(\mathcal{E})$ has a bounded t-structure.
2. For many choices of $\mathcal{E}$ and for $n < 0$, the natural map $K_n(\mathcal{E}) \longrightarrow K_{n-1}\big(\mathbf{Ac}^b(\mathcal{E})\big)$ is an isomorphism.

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And one key idea of the new paper is **to do the same**, but with $\mathcal{E}$ a stable infinity category.

And then, to show that this cannot possibly agree with the negative K-theory of $\mathbf{Ac}^b(\mathcal{E})^\heartsuit$, one argues that the negative K-theory of an abelian category has a simple chromatic structure, while $K(\mathcal{E})$ is arbitrary.

Let $\mathcal{T}$ be a model category with a bounded t–structure. Antieau, Gepner and Heller proved the following generalization of Schlichting's results:

1. If the abelian category $\mathcal{T}^\heartsuit$ is noetherian, then $K_n(\mathcal{T}) = 0$ for $n < 0$.
2. Unconditionally we have $K_{-1}(\mathcal{T}) = 0$.

If $\mathcal{A}$ is an abelian category, Schlichting's results come about by putting $\mathcal{T} = \mathbf{D}^b(\mathcal{A})$ with the standard t–structure.

Benjamin Antieau, David Gepner, and Jeremiah Heller, *K-theoretic obstructions to bounded t-structures*, Invent. Math. **216** (2019), no. 1, 241–300.

Corollary

Let X be a finite-dimensional, noetherian scheme with enough vector bundles. Assume $K_{-1}(X)$ is nonzero. Then the category $\mathbf{D}^b(\mathrm{Vect}(X))$ has no bounded t-structure.

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If $K_n(X)$ is nonzero for $n \leq -2$, then any bounded t-structure on $\mathbf{D}^b(\mathrm{Vect}(X))$ cannot have a noetherian heart.

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This can be found as Corollary 1.4 in

Benjamin Antieau, David Gepner, and Jeremiah Heller, *K-theoretic obstructions to bounded t-structures*, Invent. Math. **216** (2019), no. 1, 241–300.

Conjecture.

Let X be a finite-dimensional, noetherian scheme with enough vector bundles. The category $\mathbf{D}^b\big(\mathrm{Vect}(X)\big)$ has a bounded t-structure if and only if X is regular, in which case $\mathbf{D}^b\big(\mathrm{Vect}(X)\big) = \mathbf{D}^b_{\mathbf{coh}}(X)$.

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Theorem

Let X be a finite-dimensional, noetherian scheme with enough vector bundles. Let $Z \subset X$ be a closed subset. Let $\mathbf{D}^b_Z\big(\mathrm{Vect}(X)\big)$ be the category whose objects are the bounded complexes of vector bundles on X, whose restriction to the open set $X - Z$ is acyclic.

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The category $\mathbf{D}^b_Z\big(\mathrm{Vect}(X)\big)$ has a bounded t-structure if and only if Z is contained in the regular locus of X, in which case $\mathbf{D}^b_Z\big(\mathrm{Vect}(X)\big) = \mathbf{D}^b_{\mathbf{coh},Z}(X)$.

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The proof is in

Amnon Neeman, *Bounded t-structures on the category of perfect complexes*, Acta Math. **233** (2024), no. 2, 239–284.

Idea of the proof

Let $\mathcal{A} = \mathbf{D}^{\mathrm{perf}}_Z(X)^{\leq 0}$, and form in $\mathbf{D}_{\mathbf{qc},Z}(X)$ the t-structure with aisle

$$\mathcal{B} \quad = \quad \overline{\langle \mathcal{A} \rangle}^{(-\infty,0]} \; .$$

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The key is to prove that there exists an integer $n > 0$ with

$$\mathbf{D}_{\mathbf{qc},Z}(X)^{\leq -n} \quad \subset \quad \mathcal{B} \quad \subset \quad \mathbf{D}_{\mathbf{qc},Z}(X)^{\leq n} \ .$$

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Now let $F \in \mathbf{D}^b_{\mathbf{coh},Z}(X)$ be any object, and after shifting assume $F \in \mathbf{D}_{\mathbf{qc},Z}(X)^{\geq n+1}$.

Idea of the proof

Let $\mathcal{A} = \mathbf{D}_Z^{\mathrm{perf}}(X)^{\leq 0}$, and form in $\mathbf{D}_{\mathbf{qc},Z}(X)$ the t-structure with aisle

$$\mathcal{B} \quad = \quad \overline{\langle \mathcal{A} \rangle}^{(-\infty,0]} \ .$$

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$$\begin{array}{ccccc} D & \longrightarrow & E & \longrightarrow & F \\ \pitchfork & & & & \pitchfork \\ \mathbf{D}_{\mathbf{qc},Z}(X)^{\leq -n} & & & & \mathbf{D}_{\mathbf{qc},Z}(X)^{\geq n+1} \end{array}$$

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\overline{\langle \mathcal{A} \rangle}^{(-\infty,0]} & & & & \mathcal{A}^{\perp} \\
\cup & & & & \cup \\
\mathcal{A} & & & & \mathcal{A}^{\perp} \\
\Cup & & & & \Cup \\
E^{\leq 0} & \longrightarrow & E & \longrightarrow & E^{\geq 1}
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Thank you!