Mahler measure

A journey from Mersenne primes to special values of L-functions

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Algebra Colloquium
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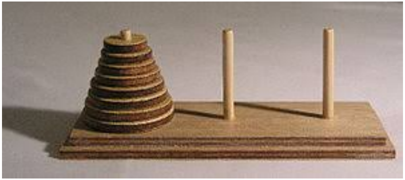
Tower of Hanoi



The objective is to move the entire stack to one of the other rods, obeying three rules.

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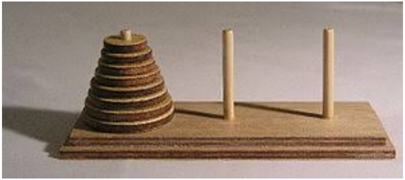

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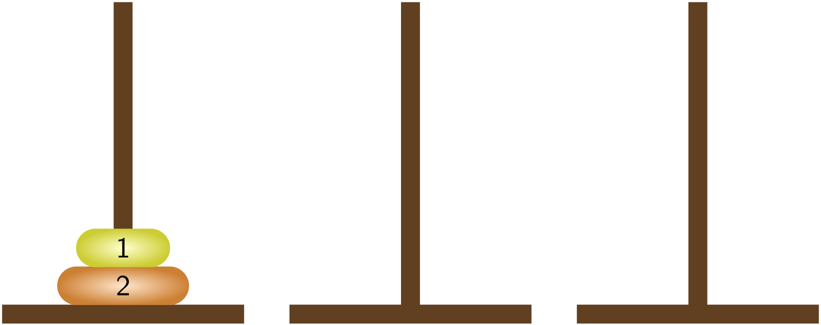
Rules

1. Only one disk may be moved at a time.
2. Each move consists of taking the upper disk from one of the stacks and placing it on top of another stack or on an empty rod.
3. No disk may be placed on top of a disk that is smaller than it.

Tower of Hanoi – 2 Discs



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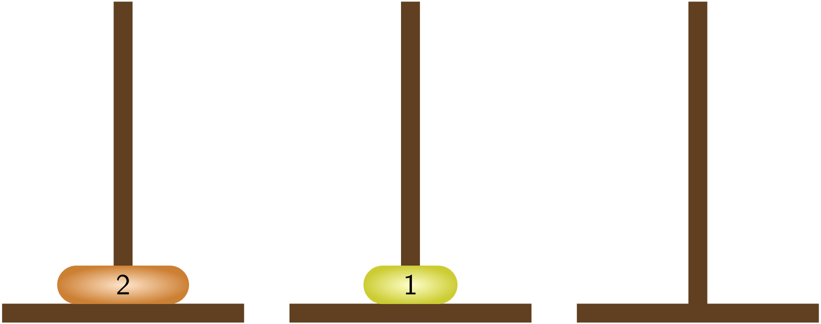

Moved disc from pole 1 to pole 2.

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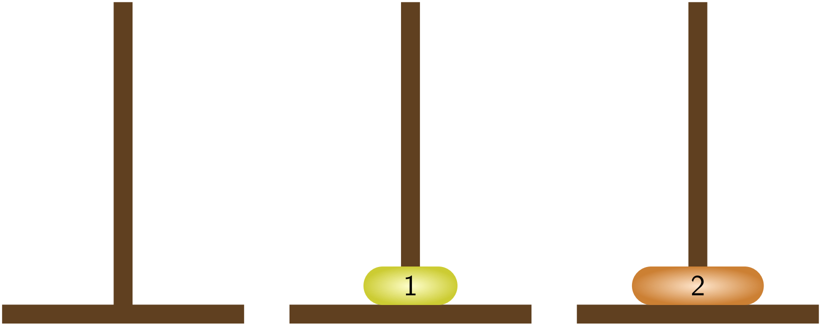
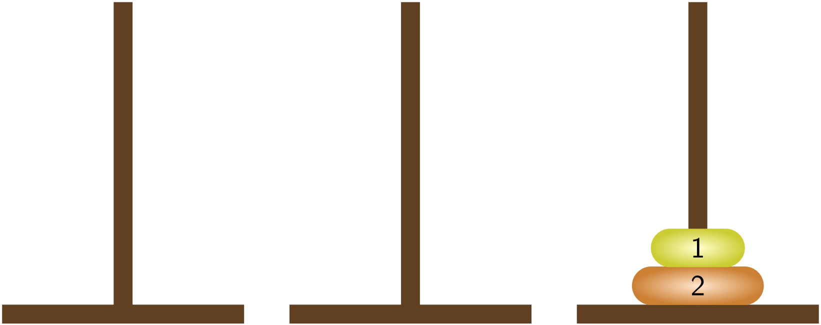


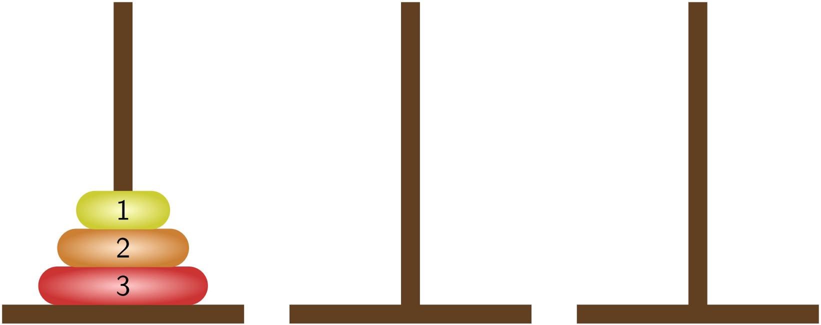
Moved disc from pole 1 to pole 3.

Tower of Hanoi – 2 Discs



Moved disc from pole 2 to pole 3.

Tower of Hanoi – 3 Discs



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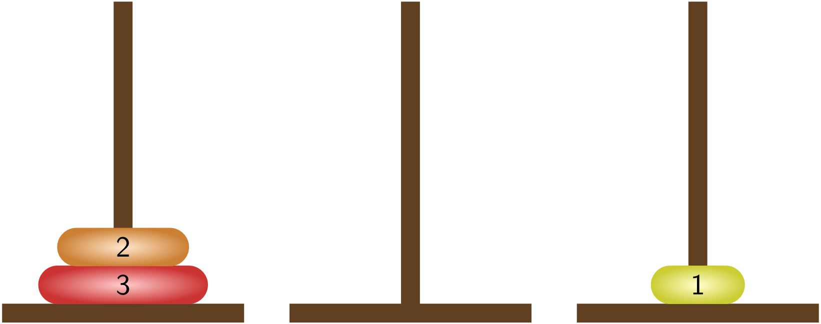

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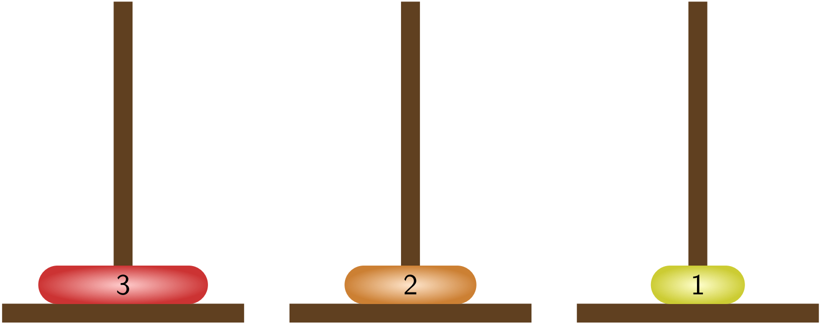
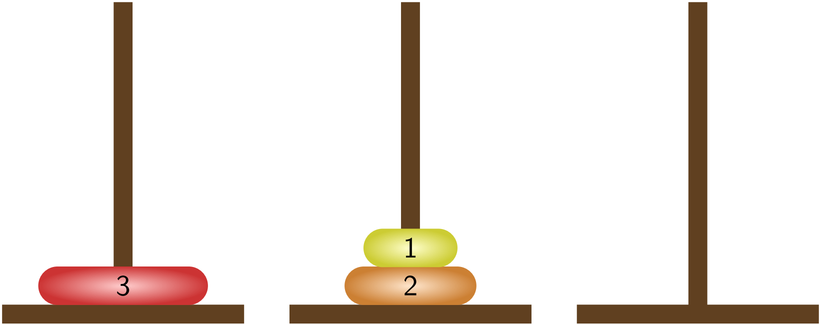


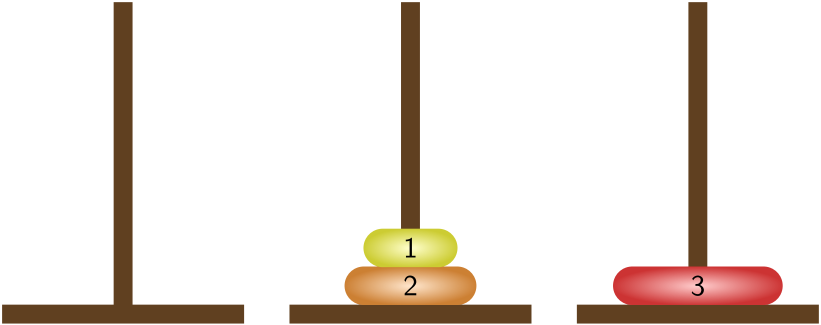
Moved disc from pole 1 to pole 2.

Tower of Hanoi – 3 Discs



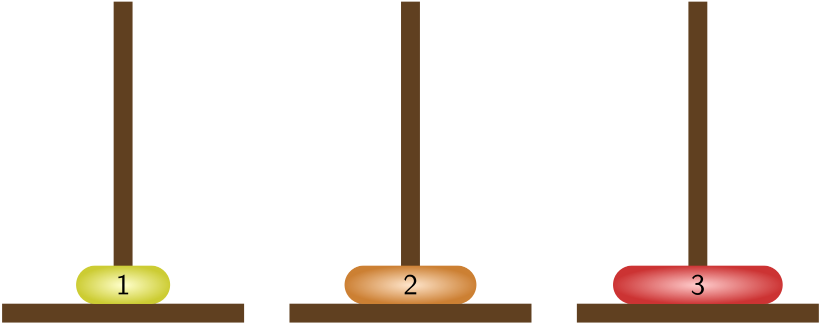
Moved disc from pole 3 to pole 2.

Tower of Hanoi – 3 Discs



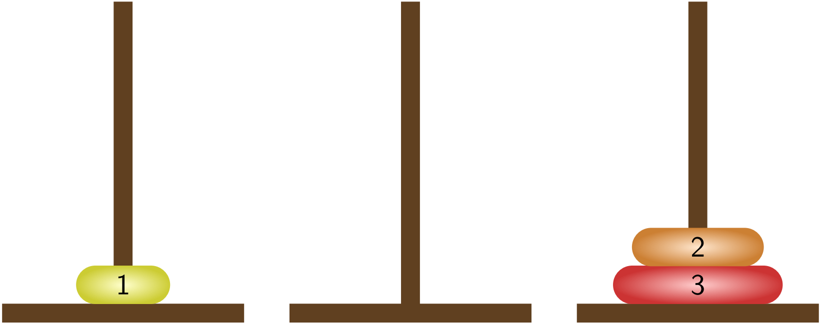
Moved disc from pole 1 to pole 3.

Tower of Hanoi – 3 Discs



Moved disc from pole 2 to pole 1.

Tower of Hanoi – 3 Discs



Moved disc from pole 2 to pole 3.

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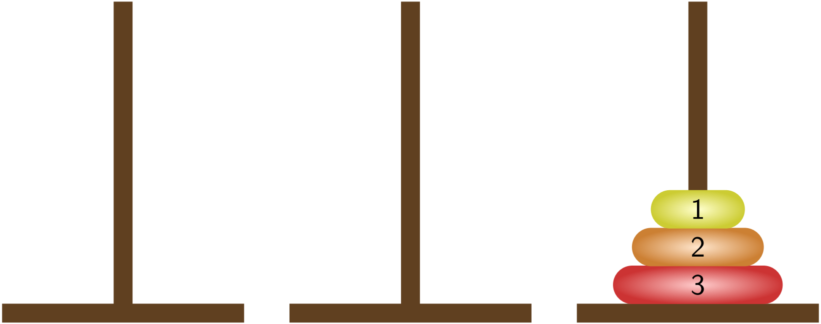
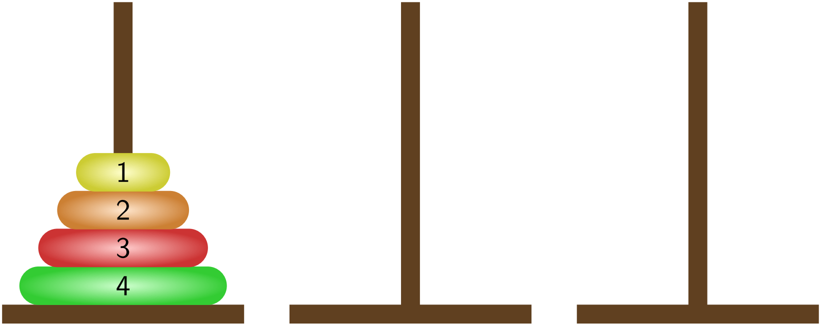


Moved disc from pole 1 to pole 3.

Tower of Hanoi – 4 Disc



Minimum number of moves?

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Definition (Marin Mersenne, early 17th Century)

A *Mersenne Number* is a number of the form $M_n = 2^n - 1$, where n is a positive integer.

Mersenne Primes

Note that if $n = ab$ (for $a, b \geq 1$),

$$2^n - 1 = 2^{ab} - 1 = (2^a - 1)\left(2^{a(b-1)} + 2^{a(b-2)} + \cdots + 1\right).$$

For example, $2^3 - 1 = 7$ is a prime, but $2^4 - 1 = 15 = 3 \times 5$ is not.

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M_n increases exponentially with n. In particular,

$$\frac{M_{n+1}}{M_n} \to 2, \text{ as } n \to \infty.$$

We expect the set $\{M_p : p \text{ is a prime}\}$ to contain *very large primes*.

A *perfect* appearance

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A *perfect number* is a positive integer that is equal to the sum of its positive proper divisors. In other words, n is perfect number if $\sigma(n) = 2n$, where $\sigma(n) = \sum_{d|n} d$.

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In the 18th century, Euler showed that all even perfect numbers are of the form $2^{p-1}M_p$, when M_p is a prime.

Finding Mersenne Primes

- Mersenne's list contains $M_2, M_3, M_5, M_7, M_{13}, M_{17}, M_{19}, M_{31}, M_{67}$ and others, which he claimed are prime numbers.

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$$\text{For } p = 23, 4 \mid (p - 3) \quad \text{and} \quad 2p + 1 = 47 \mid 761{,}838{,}257{,}287.$$

Primality test

Lucas-Lehmer test (E. Lucas 1878, D. H. Lehmer 1930)

Consider the sequence S_n for $n \geq 1$:

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Known *Mersenne primes* M_p (till now the number is ~~51~~ 52):

$$\{p = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, 110503, 132049, 216091, 756839, 859433, 1257787, 1398269, 2976221, 3021377, 6972593, 13466917, 20996011, 24036583, 25964951, 30402457, 32582657, 37156667, 42643801, 43112609, 57885161, 74207281, 77232917, 82589933, 136279841\}$$

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Largest known prime number: Mersenne prime of ~~24,862,048~~ 41,024,320 digits (*Great Internet Mersenne Prime Search (GIMPS), Luke Durant*, ~~2018~~ yesterday!!)

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- *To understand the distribution of primes.*

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- Note that $\mathrm{Res}(X-2, X^n-1) = 2^n - 1 = M_n$, where, for polynomials $P, Q \in \mathbb{C}[X]$, the *resultant* $\mathrm{Res}(P,Q)$ is defined as

$$\mathrm{Res}(P,Q) = \prod_{\alpha_i} Q(\alpha_i) = \prod_{\beta_j} P(\beta_j),$$

with $P(X) = \prod_{i=1}^{r}(X-\alpha_i)$ and $Q(X) = \prod_{j=1}^{s}(X-\beta_j)$.

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For $P \in \mathbb{Z}[X]$, can we use the sequence of integers of the form $\mathrm{Res}(P, X^n-1)$ to find large primes?

~~Total~~ Quick Recall

- An *algebraic number* $\alpha \in \mathbb{C}$ is a root of a non-zero polynomial $P_\alpha(X) = \sum_{j=0}^{d_\alpha} a_j X^j \in \mathbb{Z}[X]$, such as $\sqrt{2}$, $\sqrt[3]{\frac{1}{2}}$.

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- If $a_{d_\alpha} = 1$, then we call α an *algebraic integer*, such as $\sqrt{2}$.
- If $P_\alpha(X)$ cannot be factored into non-constant polynomials in $\mathbb{Z}[X]$ of degree $< \deg P_\alpha = d_\alpha$, then we call P_α is the *minimal polynomial* of α.

~~Total~~ Quick Recall II

- Consider $X^4 - 1$,

$$X^4 - 1 = (X-1)(X^3 + X^2 + X + 1) = (X-1)(X+1)[(X+i)(X-i)].$$

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- List of $d \mid 4$ is $\{1, 2, 4\}$. Minimal polynomial of $\zeta_1 = e^{\frac{2\pi i}{1}} = 1$ is $\phi_1(X) = (X-1)$, $\zeta_2 = e^{\frac{2\pi i}{2}} = -1$ is $\phi_2(X) = (X+1)$, and $\zeta_4 = e^{\frac{2\pi i}{4}} = i$ is $\phi_4(X) = X^2 + 1 = (X-i)(X+i)$.

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$$X^4 - 1 = \prod_{d|4} \phi_d(X).$$

- If α is a root of unity $\zeta_d^\ell = e^{\frac{2\pi i \ell}{d}}$ (such that $\gcd(\ell, d) = 1$), then the minimal polynomial is called the *d-th cyclotomic polynomial*, and denoted by $\phi_d(X)$.

A special sequence

- For $P \in \mathbb{Z}[X]$, T. Pierce (1918) and D. H. Lehmer (1932) considered $\mathrm{Res}(P, X^n - 1)$ to obtain large primes as its factors, they called it

$$\Delta_n(P) = \prod_{\alpha : P(\alpha)=0} (\alpha^n - 1).$$

A special sequence

- For $P \in \mathbb{Z}[X]$, T. Pierce (1918) and D. H. Lehmer (1932) considered $\mathrm{Res}(P, X^n - 1)$ to obtain large primes as its factors, they called it

$$\Delta_n(P) = \prod_{\alpha : P(\alpha)=0} (\alpha^n - 1).$$

- Regarding the sequence of integers Δ_n, Lehmer mentioned:

  " *These . . . are (one of) the best substitutes we have for actual formulas for prime numbers. Perhaps the most fundamental functions of this kind are special cases of* Δ_n" .

Why so special?

- Since $X^n - 1 = \prod_{d|n} \phi_d(X)$, where $\phi_d(X) \in \mathbb{Z}[X]$ is the *d-th cyclotomic polynomial*, we have

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Moral of the story: Follow "Mersenne-strategy", and restrict to $n = p$ primes and irreducible polynomials P.

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- $\Delta_n(P)$ satisfies recurrence relation.
- $\Delta_n(P)$ is periodic of period m (i.e. $\{\Delta_n(P) : n \geq 1\}$ has exactly m distinct integers) if and only if $P(X) = \phi_m(X)$.
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- For P non-cyclotomic polynomial, $|\Delta_n(P)|$ is unbounded as n grows.

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ADVANTAGE: The sequence $\{\Delta_n(P) : n \geq 1\}$ does not increase rapidly when $M(P)$ is smaller.

How small can $M(P)$ be?

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- $M(P) = \prod_{j=1}^{d} \max\{1, |\alpha_j|\} \geq 1$.
- $P(0) \neq 0$ and $M(P) = 1$ if and only if $P(X) = \prod_k \phi_k(X)$. (Kronecker's Theorem)
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Lehmer's question

Does there exists a $C > 1$, such that $M(P) \geq C$ for all non-cyclotomic polynomials?

Lower bound for $M(P)$?

A *reciprocal/symmetric polynomial* P of degree d is a polynomial satisfying $P(X) = X^d P\left(\frac{1}{X}\right)$.

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- Lehmer-Poulet (1932): For

$$P(X) = X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1,$$

$M(P) = 1.17628081\ldots$, and

$$\sqrt{\left|\frac{\Delta_{379}(P)}{\Delta_1(P)}\right|} = 1{,}794{,}327{,}140{,}357 \quad \text{is a prime.}$$

Lower bound contd.

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$$M(P) \geq 1 + c\left(\frac{\log\log d}{\log d}\right)^3.$$

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Lehmer's question is still open. The smallest known value of $M(P)$ is still due to Lehmer and Poulet.

David Boyd's approach

One of the most interesting attempts towards the resolution of Lehmer's question has been given by David W. Boyd in 1981. He considered

$$\mathcal{M} = \{M(P) : P(X) \in \mathbb{Z}[X] \setminus \{0\}\}.$$

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- If $\mathcal{M}$ is a closed subset of $[1, \infty)$, then $\mathcal{M}$ is nowhere dense, and hence 1 *is* an isolated limit point.
- Unfortunately, it is improbable that $\mathcal{M}$ is closed, as the closure of $\mathcal{M}$ may contain "special" *transcendental numbers*.

Staring at the closure

To determine the closure, we need a generalization and a theorem.

$$\mathbb{T}^n = \{(x_1, \ldots, x_n) \in \mathbb{C}^\times \times \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times : |x_1| = \cdots = |x_n| = 1\}.$$

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- For $P(X) = \prod_{j=1}^{d}(X - \alpha_j) \in \mathbb{C}[X]$,

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For $Q \in \mathbb{C}[x_1, \ldots, x_n]$, the *logarithmic Mahler measure* of Q is defined as

$$\log M(Q) = \mathrm{m}(Q) := \frac{1}{(2\pi i)^n}\int_{\mathbb{T}^n} \log|Q(x_1, \ldots, x_n)|\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}.$$

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It first appeared in a proof of Gelfond's inequality by Kurt Mahler (1962) involving a certain height of polynomials.

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$$\lim_{m_2 \to \infty} \cdots \lim_{m_n \to \infty} \mathrm{m}(Q(x_1, x_1^{m_2}, \ldots, x_1^{m_n})) = \mathrm{m}(Q(x_1, x_2, \ldots, x_n)).$$

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- $\mathcal{M}_1 = \mathcal{M} \subseteq \mathcal{M}_n \subseteq \mathcal{M}_{n+1}$, for all $n \geq 1$.
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- $\mathcal{M}_n$ are countable subsets of $[1, \infty)$, and are subsets of the set of limit points of $\mathcal{M}_1$.
- Boyd's Conjecture: $\bigcup_{n \geq 1} \mathcal{M}_n$ is closed in $[1, \infty)$, and hence nowhere dense.
- This would give an affirmative answer to Lehmer's question.

Quick detour

- *Riemann ζ-function*:

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} \qquad \text{for } \mathrm{Re}(s) > 1.$$

$\zeta(2) = \frac{\pi^2}{6}$, $\zeta(3) = 1.202056903\ldots$ is irrational (Apéry).

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- *L-function* an arithmetic "object" X expressed as

$$L(X, s) = \sum_{n \geq 1} \frac{a_n(X)}{n^s} \qquad \text{for } \mathrm{Re}(s) > \sigma,$$

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- Special values of $L(X, s)$ and (logarithmic) Mahler measure belong to a set of special kinds of numbers called *periods*, which have applications in transcendental number theory and algebraic geometry.

What numbers appear in $\log \mathcal{M}_n$?

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- Deninger (1996): $\mathrm{m}(P)$ as certain integral of a differential n-form over a algebraic n-cycle, and conjectured

$$\mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + 1\right) \stackrel{?}{=} \frac{15}{4\pi^2} L(E_{15}, 2),$$

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  later proved by Rogers-Zudilin (2010).
- Boyd (1998): For some families of polynomials $P_k(x,y)$, with parameter k,

$$\mathrm{m}(P_k) \stackrel{?}{=} \frac{r_k N(k)}{4\pi^2} L(E_{N(k)}, 2),$$

  where $k^2 \in \mathbb{Z}$, $r_k \in \mathbb{Q}$ and $E_{N(k)}$ is an elliptic curve associated to the polynomial P_k.

Examples contd.

- Rodriguez-Villegas (1997): Proved Boyd's conjecture for

$$Q_k := Q_k(x,y) = x + \frac{1}{x} + y + \frac{1}{y} + k,$$

when $k = 2\sqrt{2}, 4\sqrt{2}$.

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- Lalín (2010): Proved Boyd's conjecture for $k = 5$.
- Brunault (2015): Proved Boyd's conjecture for $k = 12$.

Generalized Mahler measure

In a different direction, Cassaigne-Maillot (2000) generalized the result of Smyth (1981), and explicitly computed $\mathrm{m}(ax + by + c)$ in terms of the volume of a certain hyperbolic triangle depending on $|a|$, $|b|$ and $|c|$, for $a, b, c \in \mathbb{C}^*$.

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Definition

The *generalized Mahler measure* of a Laurent polynomial $P \in \mathbb{C}(x_1, \ldots, x_n)^*$ is defined as

$$\mathrm{m}_{a_1,\ldots,a_n}(P) := \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n_{\mathbf{a}}} \log |P(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n},$$

where $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n_{>0}$ and

$$\mathbb{T}^n_{\mathbf{a}} := \{(x_1, \ldots, x_n) \in \mathbb{C}^* \times \mathbb{C}^* \times \cdots \times \mathbb{C}^* : |x_1| = a_1, \ldots, |x_n| = a_n\}.$$

Result

Recall the family $Q_k(x,y) = x + \frac{1}{x} + y + \frac{1}{y} + k$, and consider

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- R.(2024): Given $a, b > 0$, if $Q_k(x,y) \neq 0$ for any $(x,y) \in \mathbb{T}^2_{a,b}$, then

$$\mathrm{m}_{a,b}(Q_k) = \mathrm{m}(Q_k(ax, by)) = \mathrm{m}(Q_k) \quad \text{or} \quad \max\{\log a, \log b\}.$$

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Recall the family $Q_k(x,y) = x + \frac{1}{x} + y + \frac{1}{y} + k$, and consider

$$\mathbb{T}^2_{a,b} = \{(x,y) \in \mathbb{C}^\times \times \mathbb{C}^\times : |x| = a, |y| = b\}.$$

- R.(2024): Given $a, b > 0$, if $Q_k(x,y) \neq 0$ for any $(x,y) \in \mathbb{T}^2_{a,b}$, then

$$\mathrm{m}_{a,b}(Q_k) = \mathrm{m}(Q_k(ax, by)) = \mathrm{m}(Q_k) \quad \text{or} \quad \max\{\log a, \log b\}.$$

- This shows that, under the non-vanishing condition, shrinking or extending the integration torus either does not change the Mahler measure or the change is linear.

contd.

- This provides evidence for Boyd's conjecture of a large number of polynomials. For example, for $a \in \left(\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}+1}{2}\right)$,

$$\mathrm{m}_{a,a}(Q_{2i}) = \mathrm{m}_{a,\sqrt{a}}(Q_{2i}) = \mathrm{m}(Q_{2i}) = \frac{40}{4\pi^2} L(E_{40}, 2),$$

where the last equality is due to Mellit (2011).

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- Similar methodology of the proof extends the result for any n-variable polynomial with complex coefficients.

- R. (2024): explicit expression of $\mathrm{m}_{a,b}(Q_4)$ in terms of hyperbolic volumes for all $a, b > 0$.

Some final destinations

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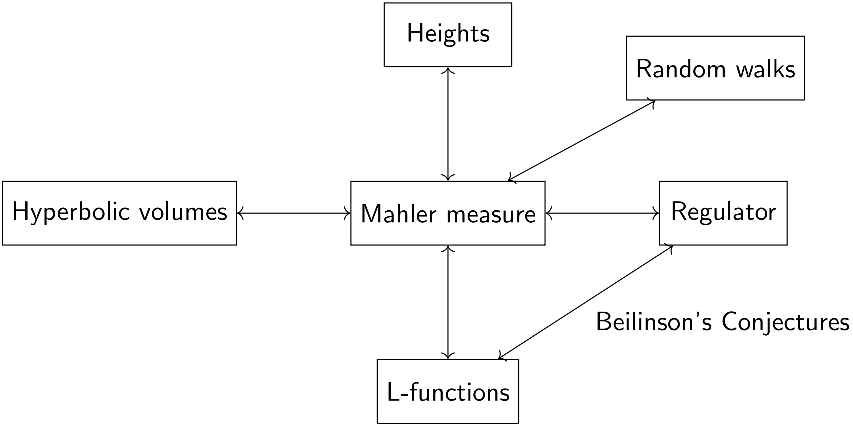
- Mahler measures are also considered as certain height functions of polynomials or varieties defined by polynomials. Different generalizations of Mahler measures are considered based on this fact, such as *Dynamical Mahler measure* etc.
- Certain changes of the integration domain, affect the Mahler measure significantly. For example, replacing unit torus by unit disc results in a negative answer for Lehmer's question in the respective setting.

Mahler measure, a symphony!



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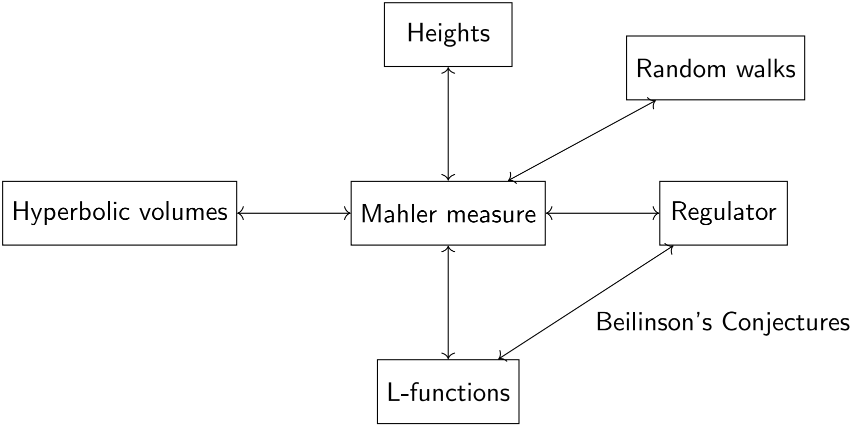

A symphony must be like the world. It must contain everything.

— Gustav Mahler.

Thank you for your attention!