The pp-constructability poset

(Recent results & Missing pieces)

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What is Universal Algebra?

– Model Theory without relational symbols

Basic object: $(\underbrace{A}_{\text{set}} ; f_1, f_2, \ldots)$ $\quad f_1 : A^{ar(f_1)} \to A$

e.g., a group is an algebra $\mathbb{G} = (G ; \underbrace{\cdot}_{\text{binary}}, \underbrace{{}^{-1}}_{\text{unary}}, 1)$ — nullary / constant

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e.g., a group is an algebra $\mathbb{G} = (G;\ \cdot\ ,\ ^{-1},\ 1)$ nullary /constant

binary $\quad$ unary

- Generalization of permutation group theory

We consider functions which have arity ≥ 1.

(Operation clones $\longleftrightarrow$ Permutation groups)

Main Goal

- Describe all algebras (up to "something")

  Good understanding of 2-element algebras ✓

  We know something non-trivial about n-element algebras, $n \geq 3$. 😐

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- Tools / Theories

Foundation	Birkhoff's HSP Theorem; Free algebras; ...	
Commutator Theory	70'	[Smith; Freese, McKenzie; ...]
Tame Congruence Theory	80'	[Hobbie, McKenzie; ...]
Congruence Varieties	80'+	[Garcia, Taylor; Kearnes, Kiss; ...]
Bulatov Theory	2000+	[Bulatov]
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work well for finite

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signature : $\Sigma : \{ f_1, f_2, \dots \}$; arity : $\Sigma \to \mathbb{N}_0$

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Def: An operation $f : A^n \to A$ preserves a relation $R \subseteq A^k$ if

$$\forall a_1, \dots, a_n \in R : \begin{pmatrix} f(a_1^1, \dots, a_n^1) \\ \vdots \\ f(a_1^k, \dots, a_n^k) \end{pmatrix} \in R.$$

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: $R \leq \mathbf{A}^k$ (R is a subpower of $\mathbf{A}$, i.e., $R \in SP(\mathbf{A})$)

(Operation) Clones

Questions:

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- signature-free algebras

If $\mathbf{A} = (A; \ldots)$, $\mathbf{B} = (A; \ldots)$ [with possibly different signatures] have the same set of term operations, they can be considered "equal".

Reason: many properties are determined by their term-clone

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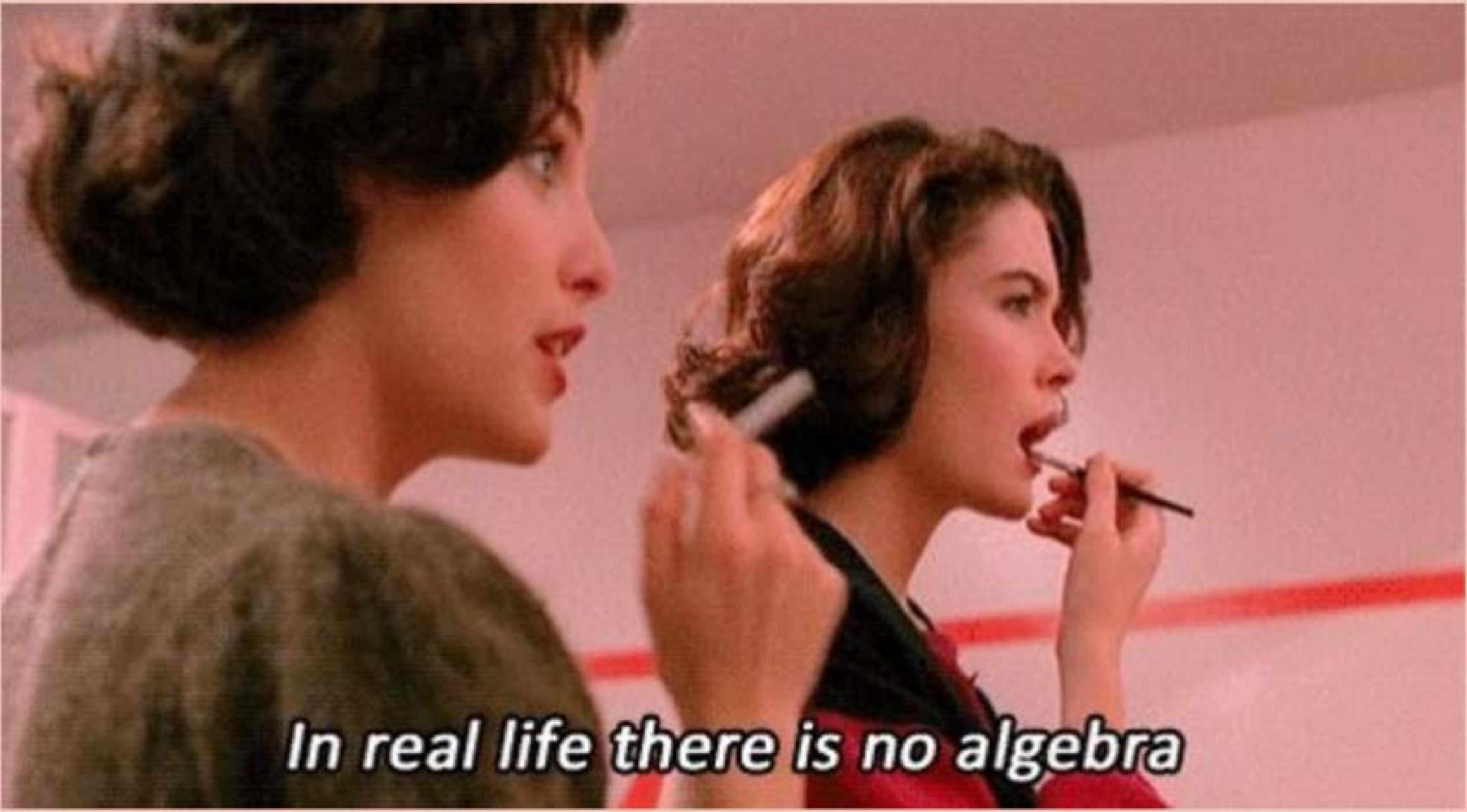
② Where do clones show up in real life?


In real life there is no algebra

(Operation) Clones

permutation group

in "real" life

Aut(...)

set of bijections $A \to A$
containing the inverse, identity,
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(Operation) Clones

more general $\rightarrow$

permutation group $\rightsquigarrow$ transformation monoid

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Def: a clone over a (finite) set A is a set of operations $\mathcal{A}$ over A s.t.

- $\mathcal{A}$ contains all projections $\rightsquigarrow$ $(\pi_i^n : A^n \to A \ ; \ \pi_i^n(a_1, \dots, a_n) = a_i)$
- $\mathcal{A}$ is closed under composition $\rightsquigarrow$ f: n-ary operation in $\mathcal{A}$; $g_1, \dots, g_n$: k-ary op. in $\mathcal{A}$

$$\rightarrow f(g_1, \dots, g_n)(a_1, \dots, a_k) := f(g_1(a_1, \dots, a_k), \dots, g_n(a_1, \dots, a_k)) \in \mathcal{A}$$

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F: set of operations $\rightarrow$ $\langle F \rangle$: the clone generated by F

(Operation) Clones

Examples:

- projections

  $= \langle \emptyset \rangle$: clone over any set that only contains projs

- all operations

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  $\mathrm{Clo}(\{0,1\}; \wedge) \rightsquigarrow$ product of variables

  $x_3 \wedge x_5 \wedge x_6$

  $\mathrm{Clo}(\{0,1\}; \wedge, \vee) \rightsquigarrow$ all monotone idempotent boolean operations

  $(x_1 \wedge x_3) \vee (x_4 \wedge x_7 \wedge x_5)$

  $f(x, \ldots, x) = x$

  3-ary majority

  $\mathrm{Clo}(\{0,1\}; d_3) \rightsquigarrow$ all monotone & self-dual boolean op.

  f preserves $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

  $\mathrm{Clo}(\{0,1\}; m) \rightsquigarrow$ sums of odd number of variables (mod 2)

  $(x+y+z) \bmod 2 =$ 3-ary minority

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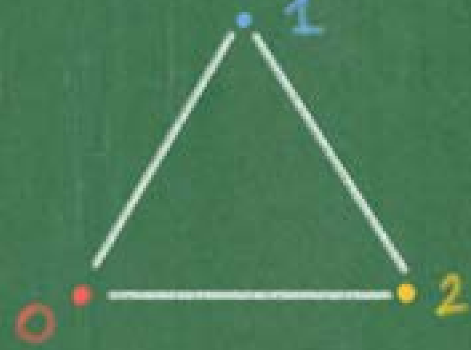
Example:

$$\mathbb{K}_3 := (\{0,1,2\}; \neq, \{0\}, \{1\}, \{2\})$$

$$\mathrm{Pol}(\mathbb{K}_3) = \langle \emptyset \rangle = \text{projections}$$



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: $\mathrm{Inv}(F)$ is a relational clone, $\forall F$ set of operations over A (finite).

A Galois connection for clones

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Examples: $\langle m, \wedge \rangle = \mathrm{Pol}(\{0,1\}; \{0\}, \{1\})$ all idempotent boolean oper.

$\langle \wedge, \vee \rangle = \mathrm{Pol}(\{0,1\}; \leq, \{0\}, \{1\})$

$\langle d_3 \rangle = \mathrm{Pol}(\{0,1\}; \leq, \neq, \{0\}, \{1\})$

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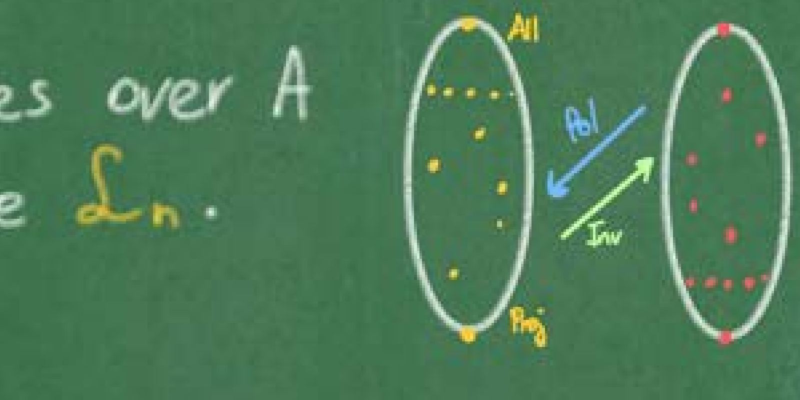
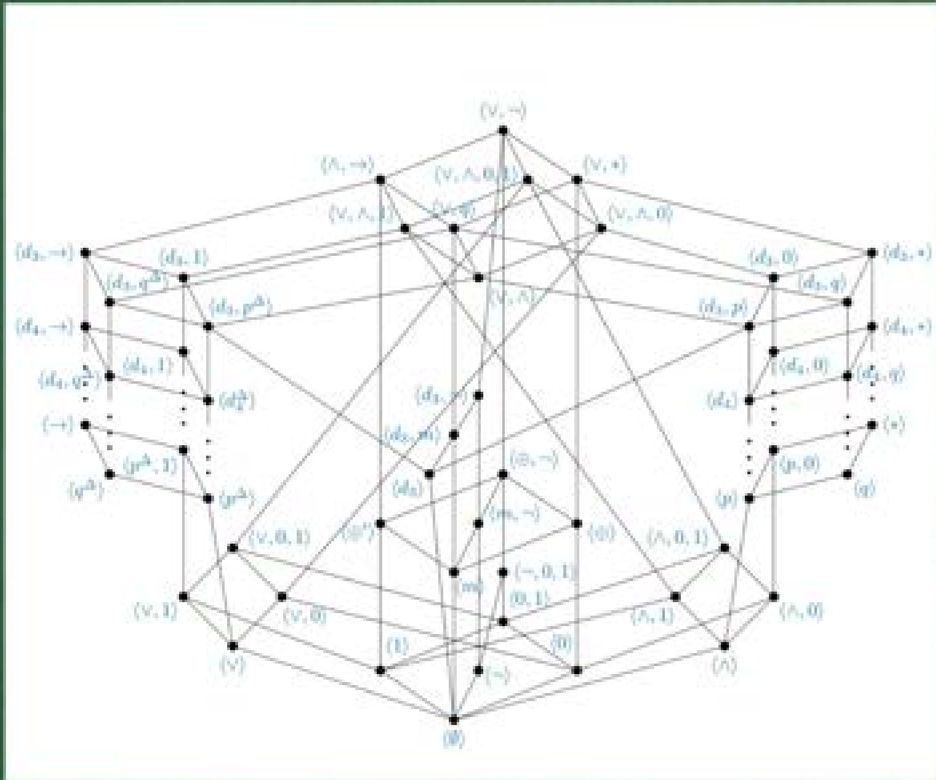
: A: set s.t. $|A| = n$. The set of all clones over A ordered by inclusion forms a lattice $\mathcal{L}_n$.



Post's Lattice (and more)



$\mathcal{L}_2$: clones over $\{0,1\}$, up to $\subseteq$

- Fully described by Post '21
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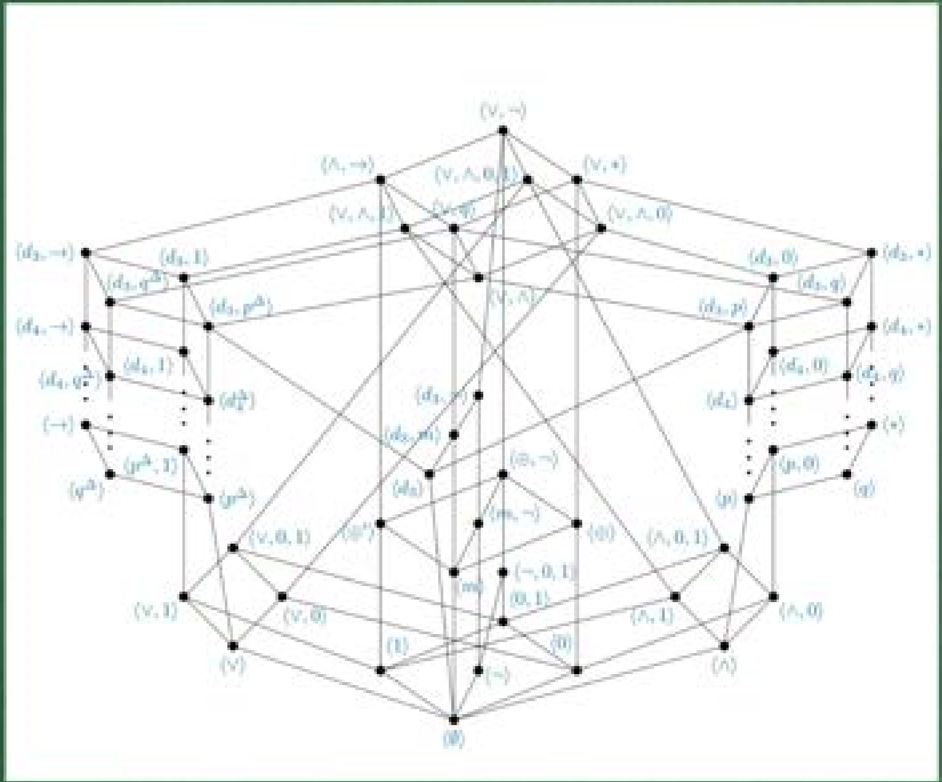

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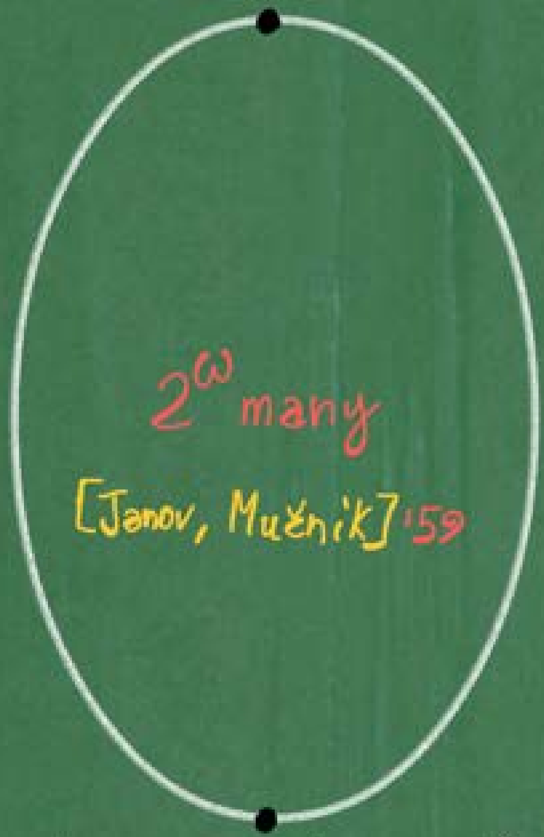

$\mathcal{L}_n$: clones over $\{0,1,..,n-1\}$, up to $\subseteq$
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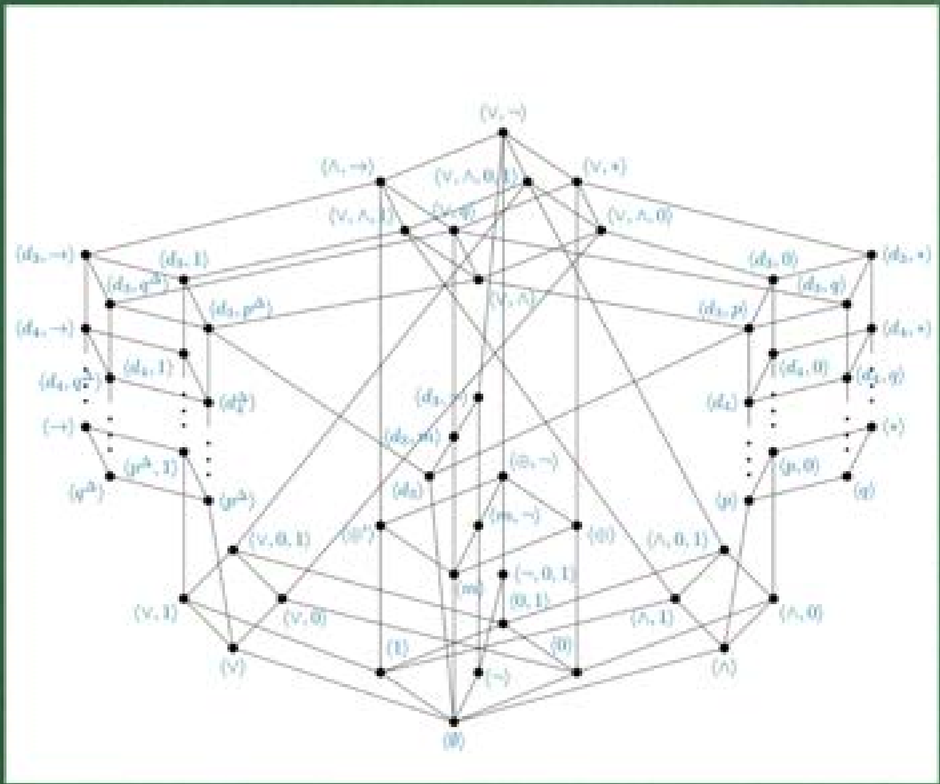

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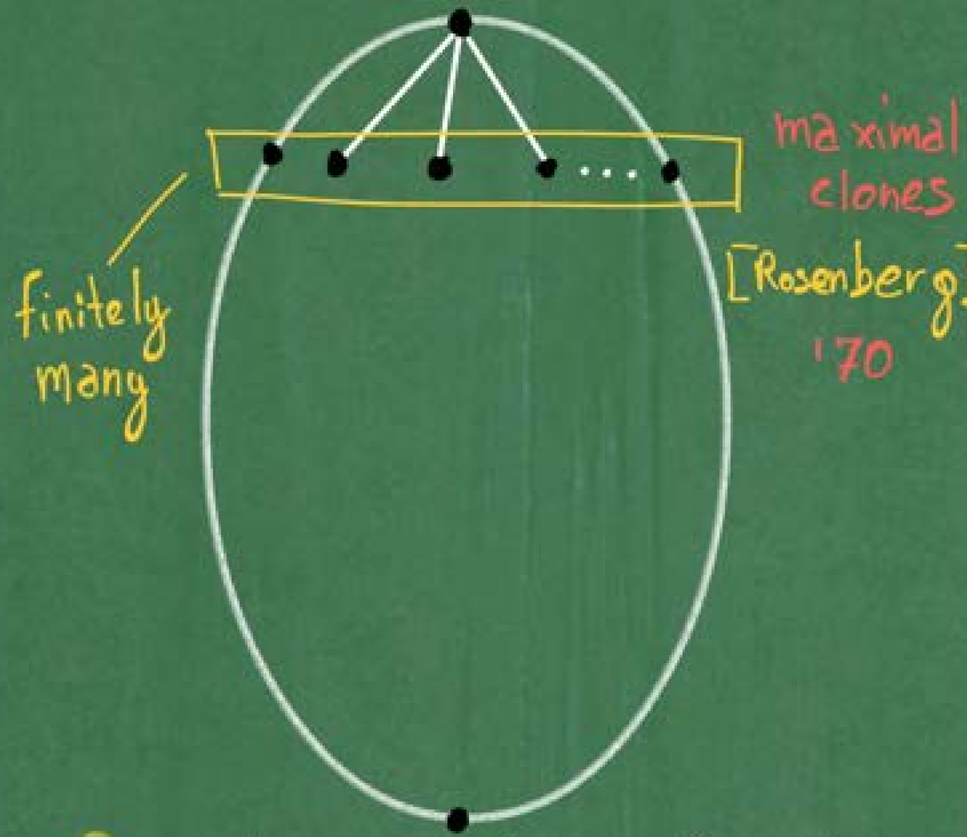

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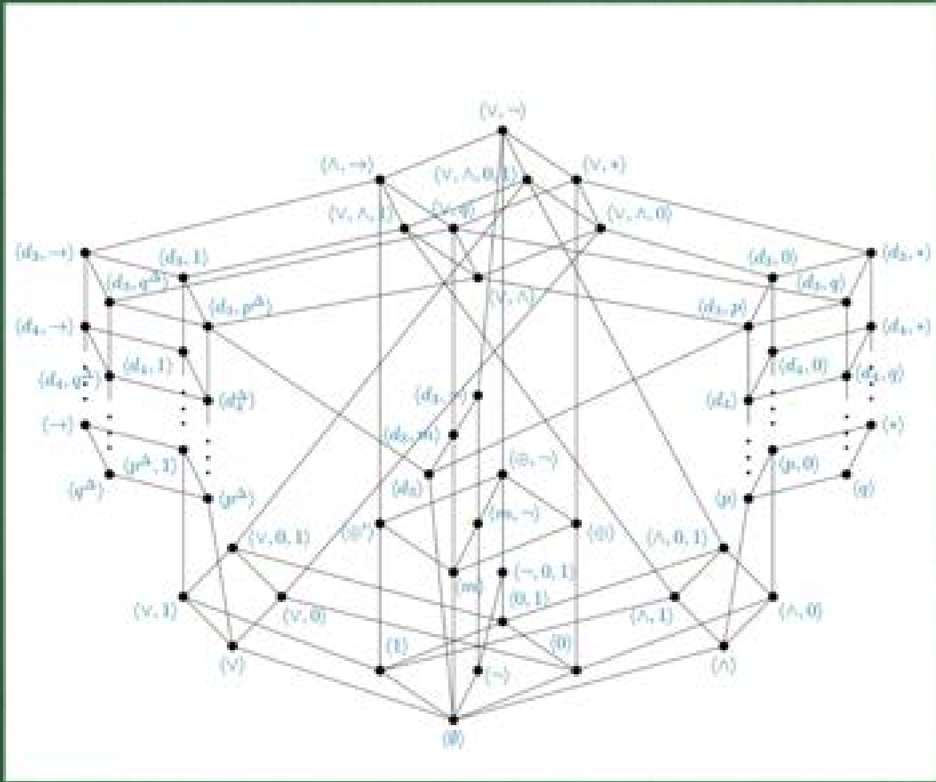

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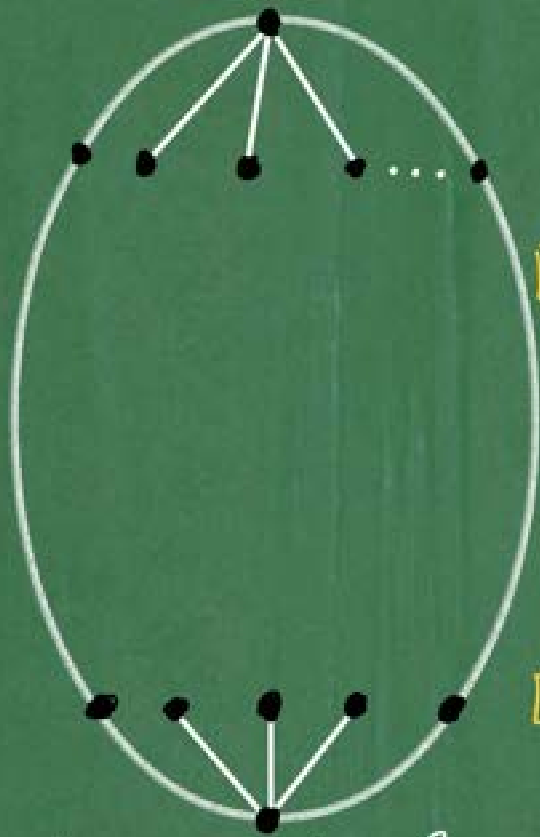

maximal clones [Jablonskij] '54

minimal clones [Csákany] '83

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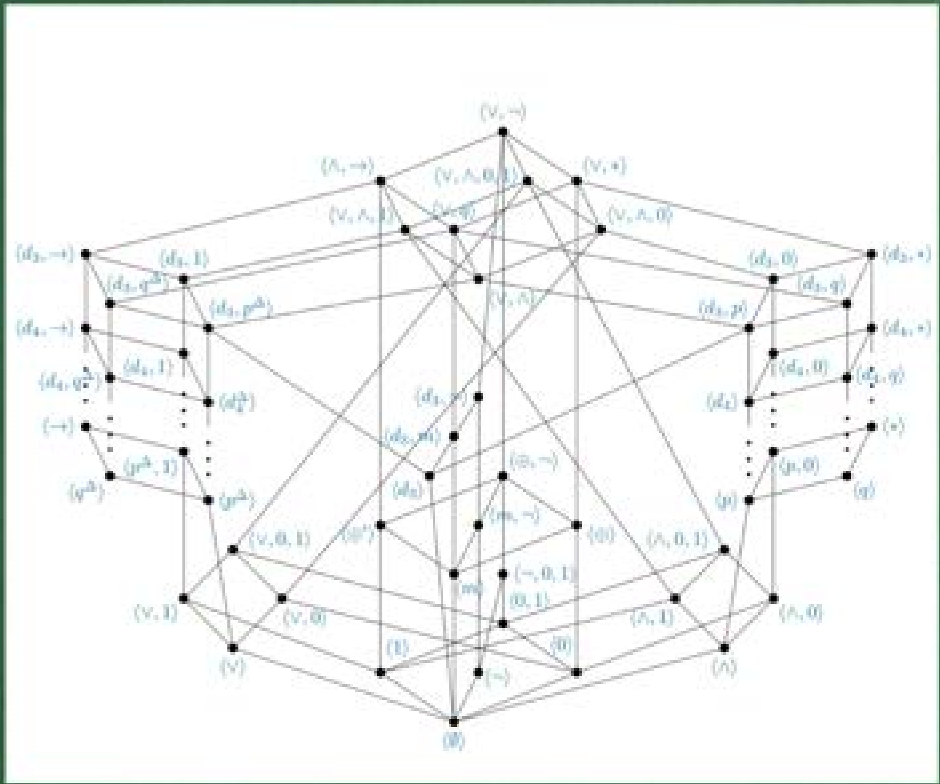

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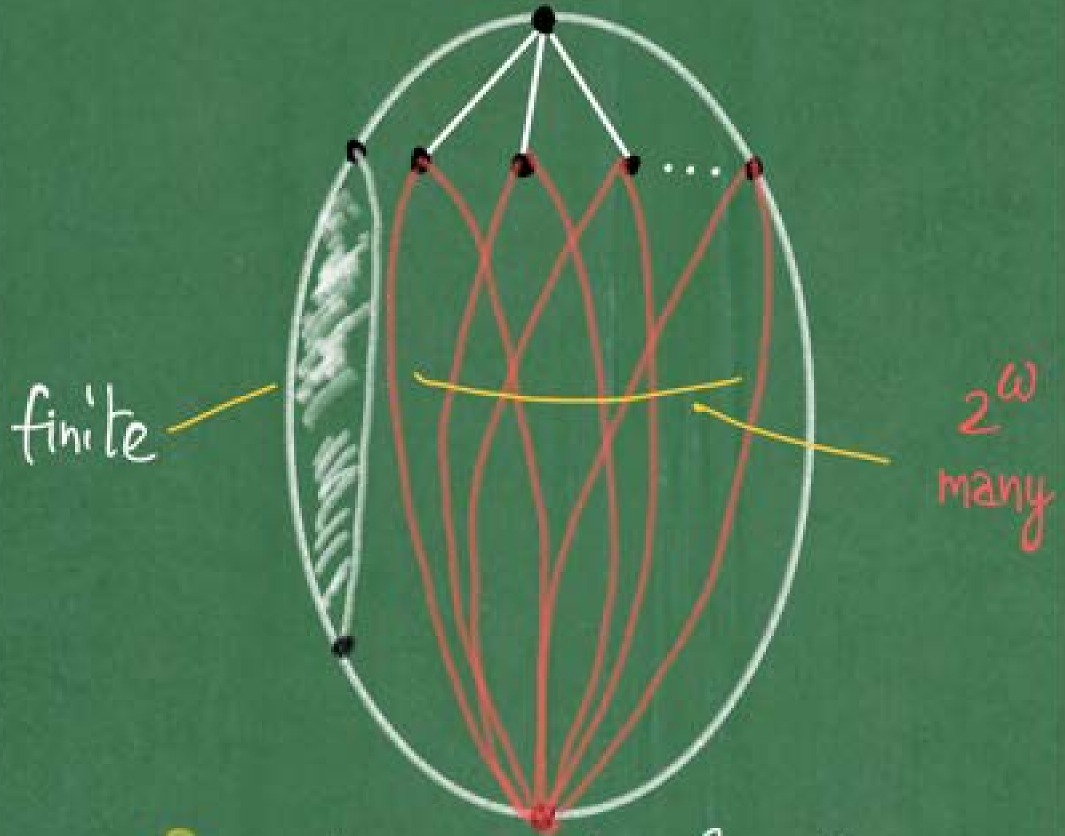

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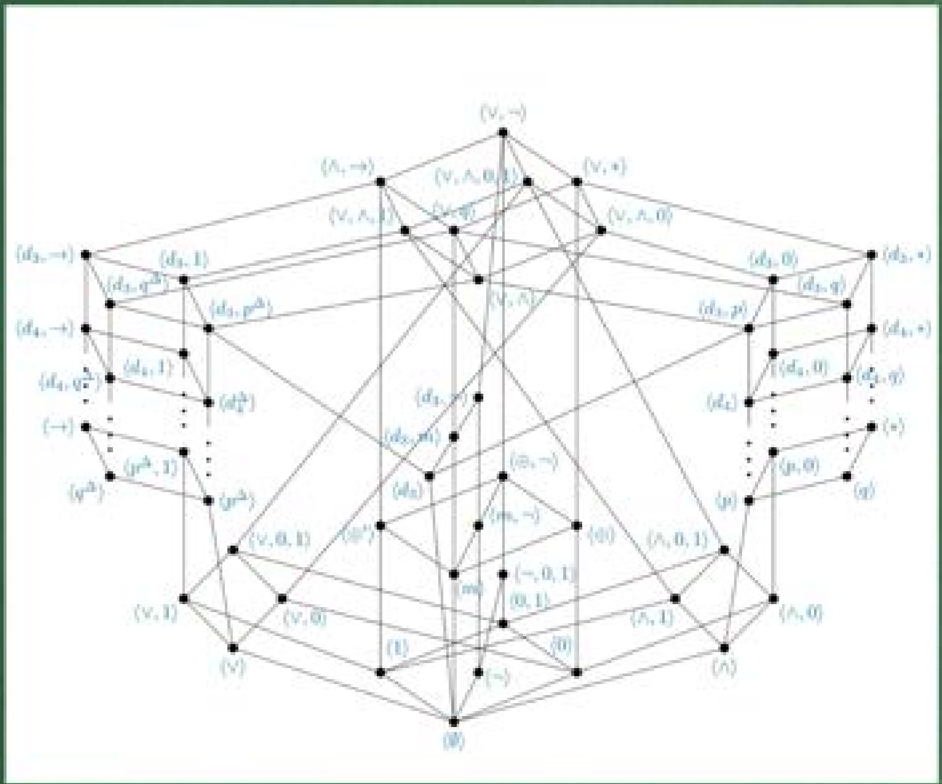

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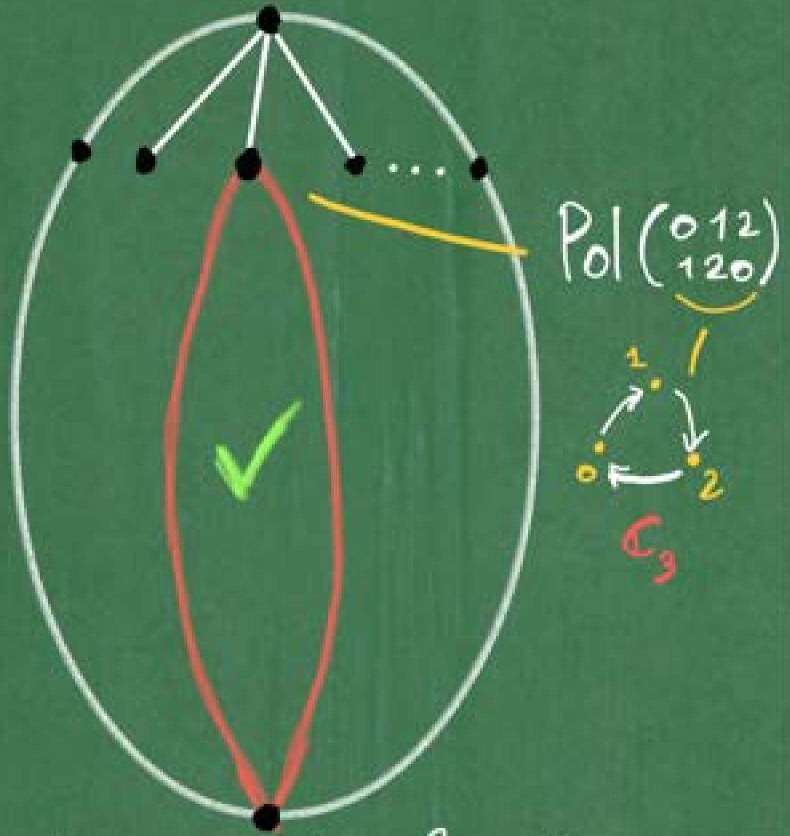

$\mathcal{L}_3$: clones over $\{0,1,2\}$, up to $\subseteq$

- D. Zhuk completely described the sublattice of clones $\subseteq \mathrm{Pol}(\mathbb{C}_3)$ 2015

What next?

Recall: $\mathrm{Pol}(\mathbb{A}) \subseteq \mathrm{Pol}(\mathbb{B})$ iff $\mathbb{A}$ pp-defines $\mathbb{B}$.

rel. structures over
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① clone homomorphisms

"Some clones are not so different" e.g.: $\langle \vee \rangle$ & $\langle \wedge \rangle$

Def: $\mathcal{C}, \mathcal{D}$ clones over some finite univ. ; a clone hom. is a mapping $\xi : \mathcal{C} \to \mathcal{D}$ preserving arities and s.t.

- preserves projections
- preserves composition $\xi(f(g_1, \ldots, g_n)) = \xi(f)(\xi(g_1), \ldots, \xi(g_n))$

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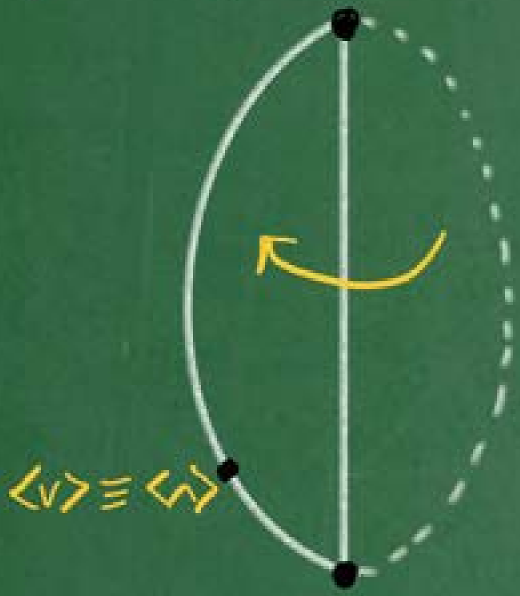

$\mathcal{L}_2$ "after clone hom."

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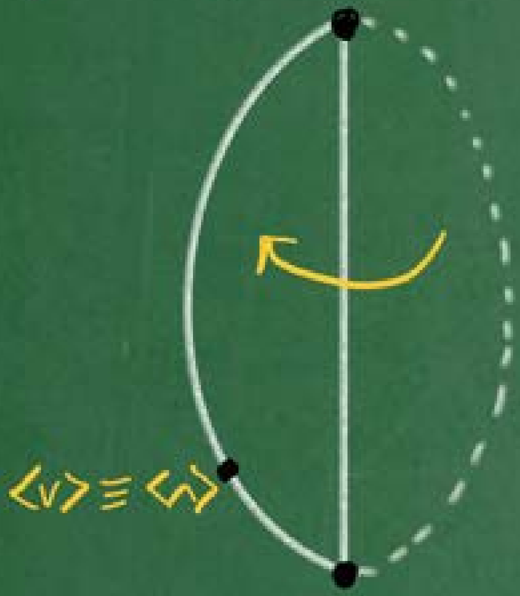

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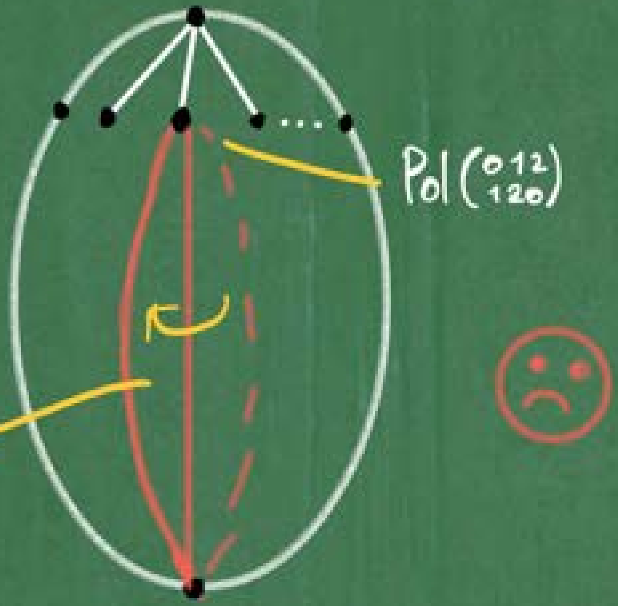

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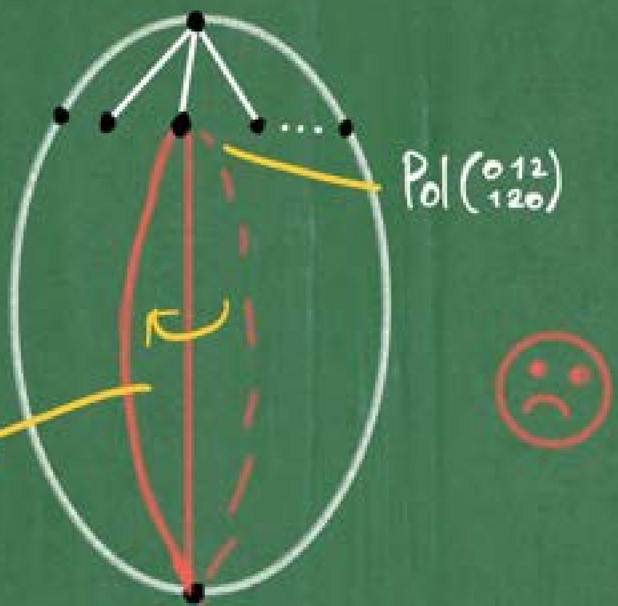

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• It has good properties though:

- can compare clones over different universes
- it preserves complexity of CSPs

⋮

We need more power!

Def: For a fixed $\mathbb{A}$ over some fin. signature τ, $\mathrm{CSP}(\mathbb{A})$ is the membership problem of $\{\mathbb{S} \mid \mathbb{S} \text{ is a } \tau\text{-struct. and } \mathbb{S} \to \mathbb{A}\}$

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$\exists$ map $h : A \to B$ st. $\forall R \in \tau$:
$\bar{a} \in R^{\mathbb{A}} \Rightarrow h(\bar{a}) \in R^{\mathbb{B}}$

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<u>Example</u>: $\mathrm{CSP}(\mathbb{K}_3) \leftrightarrow$ deciding whether a given graph is 3-colorable

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(in logspace)

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(in logspace)

- Common ways to "manipulate" an operation:
  - identify variables
  - permute variables
  - add "dummy" variables

We need more power!

Def: For a fixed $\mathbb{A}$ over some fin. signature τ, $CSP(\mathbb{A})$ is the membership problem of $\{\mathbb{S} \mid \mathbb{S} \text{ is a } \tau\text{-struct. and } \mathbb{S} \to \mathbb{A}\}$

Example: $CSP(\mathbb{K}_3) \leftrightarrow$ deciding whether a given graph is 3-colorable

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Def: f n-ary operation; $\sigma : \{1, \ldots, n\} \to \{1, \ldots, r\}$. We denote by f^σ the r-ary operation $f^\sigma(x_1, \ldots, x_r) := f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$

minor of f

The Wonderland of reflections

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② minion homomorphisms
(a.k.a minor-preserving maps)

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Def: a map $\xi : \mathcal{C} \to \mathcal{D}$ preserving arieties and s.t.

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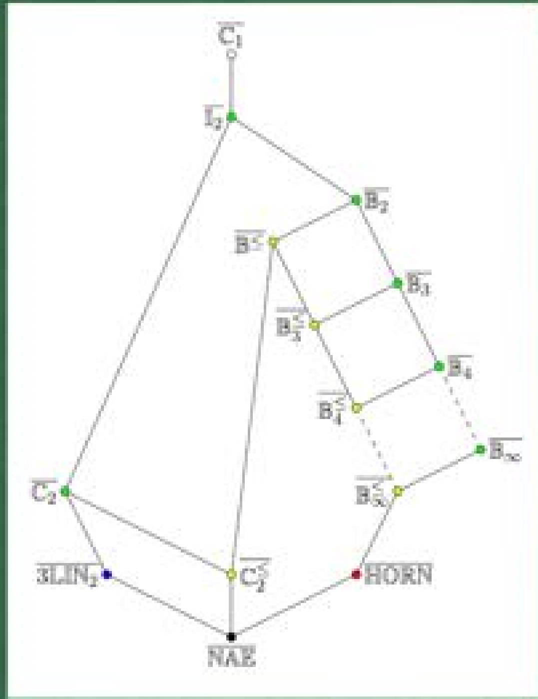
Thm [Barto, Opršal, Pinsker]: $Pol(\mathbb{A}) \preccurlyeq_m Pol(\mathbb{B}) \iff \mathbb{A}$ pp-constructs $\mathbb{B}$

How powerful is this notion?



$\mathcal{L}_2$ "after minion hom."

Bodirsky; V.

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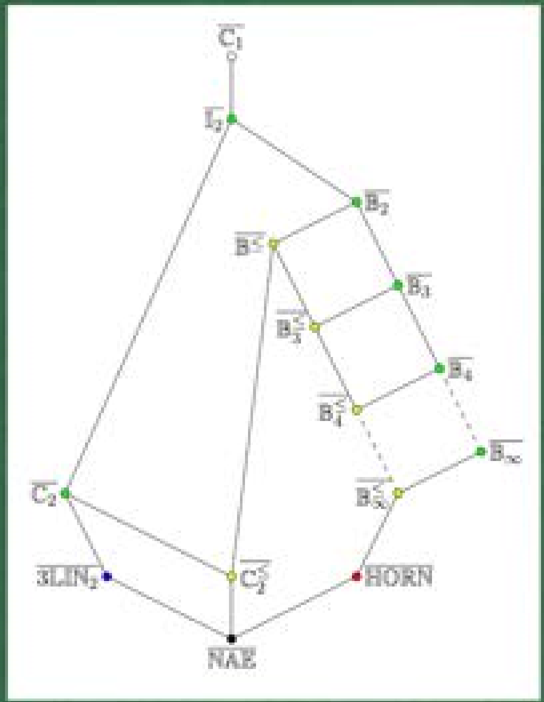

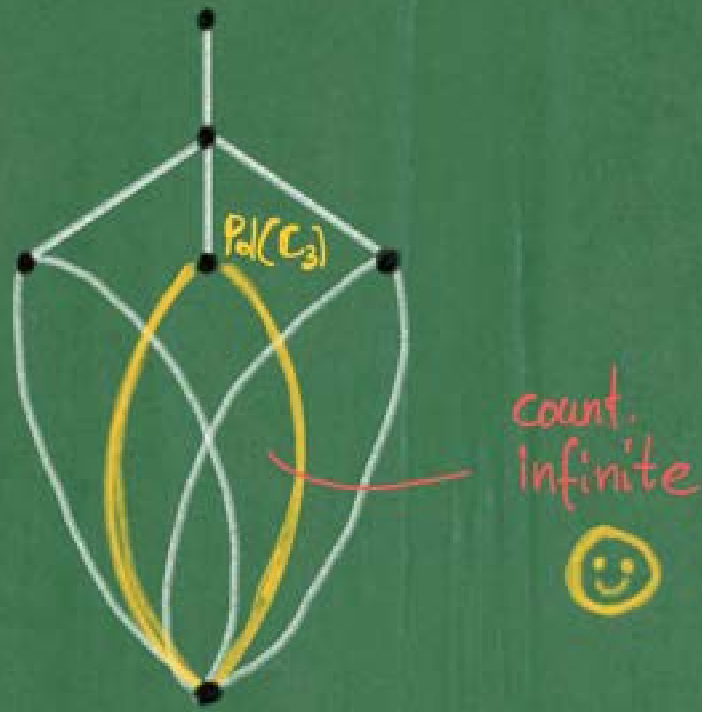

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$\mathfrak{L}_3$ "after minion hom."

Bodirsky ; V. ; Zhuk

Recent results & Missing pieces

- Clones over $\{0,1,2\}$ defined by binary relations (up to $\leq_m$)
  Barto, Olšák, Starke, V., Zahálka, Zhuk

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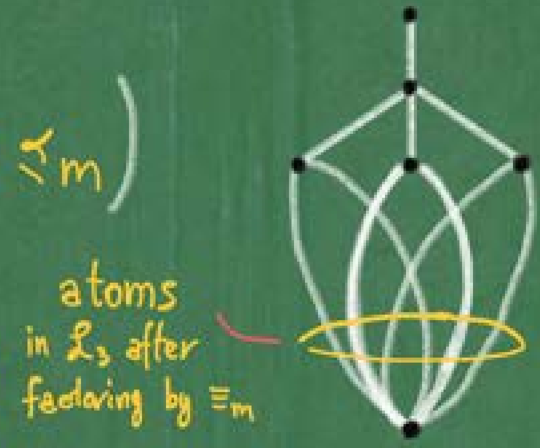

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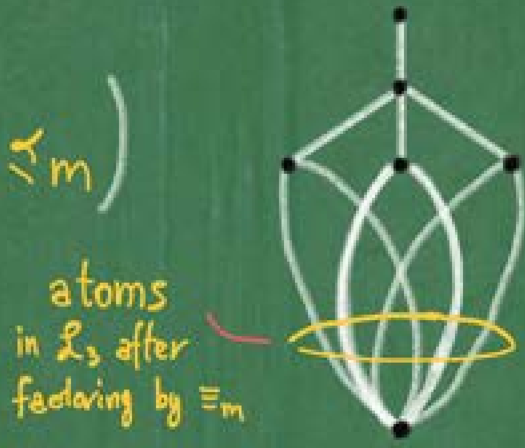
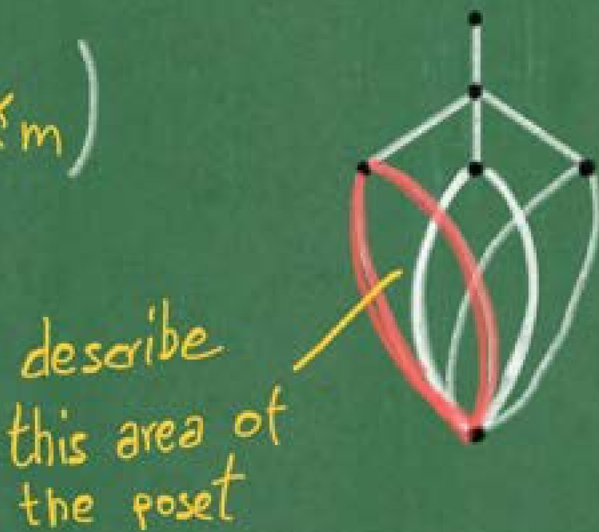


- Mal'cev clones over $\{0,1,2\}$ (up to $\precsim_m$)

  Fioravanti, Kompatscher, Rossi, V.



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- Multisorted Boolean Clones Determined by bin. rel. (up to $\equiv_m$)

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Nobody ... yet

pp-constructions

Def: $\mathbb{B}$ is a pp-power of $\mathbb{A}$ if $\mathbb{B}$ is isomorphic to a structure $\mathbb{P}$ s.t.

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Claim: $\mathbb{A}$ pp-constructs $\mathbb{B}$ $\rightsquigarrow$ consider $\mathbb{P} = (\{0,1\}^2; \Phi_{\leq}, \Phi_R, \Phi_0, \Phi_1)$

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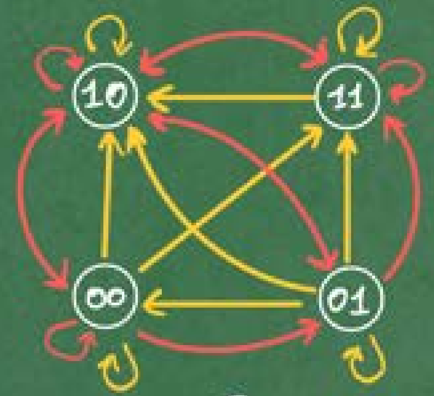

$\mathbb{S}$

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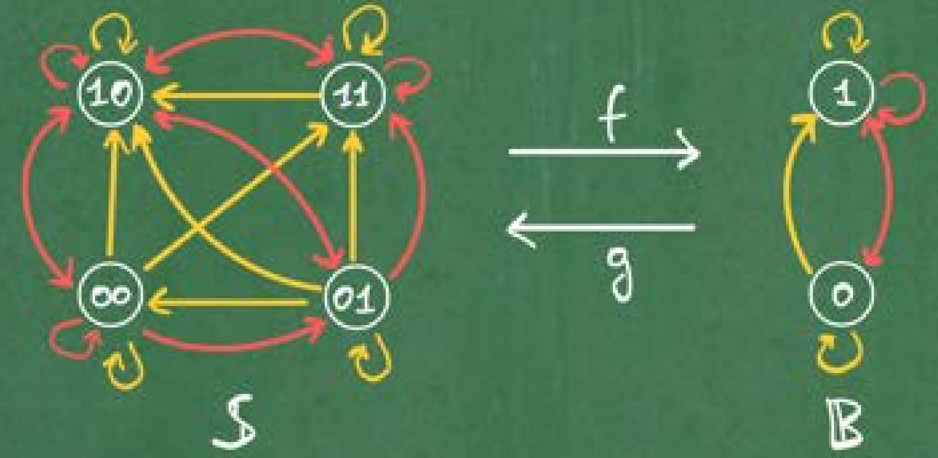

$f: \begin{array}{l} (0,1) \mapsto 0 \\ (0,0), (1,0), (1,1) \mapsto 1 \end{array}$ $\qquad$ $g: \begin{array}{l} 0 \mapsto (0,1) \\ 1 \mapsto (1,0) \end{array}$


Thank you !